September 1 Math 2306 sec. 52 Spring 2023

Section 4: First Order Equations: Linear & Special

We're considering two types of first order differential equations in this section. A **first order linear** equation is one of the form

$$\frac{dy}{dx} + P(x)y = f(x), \tag{1}$$

Let's recall the solution process, and then consider equations known as **Bernoulli** differential equations.

Solution Process 1^{st} Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx$$

$$y(x) = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Bernoulli Equations

Bernoulli 1^{st} Order Equation

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: A Bernoulli equation looks like a linear one at first glance. However, since $n \neq 0, 1$ a Bernoulli equation is necessarily **nonlinear**.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{2}$$

We'll solve (2) by using a change of variables

$$u = y^{1-n}.$$

The new variable u will satisfy a linear equation which we will solve and substitute back $y = u^{\frac{1}{1-n}}$.

We'll derive an equation for u. Let's find $\frac{dy}{dx}$.

$1-n-1=-n$

$$u = y^{1-n} \Rightarrow \frac{d}{dx}u = \frac{d}{dx}y^{1-n} \Rightarrow \frac{du}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx}$$

Divide by $1-n$ and multiply by y^n

$$\frac{dy}{dx} = \frac{1}{1-n}y^n\frac{du}{dx}$$

Sub into the ODE

$$\frac{1}{1-n}\, y^n \frac{du}{dx} + P(x)\, y = f(x)\, y^n$$

Multiply by $1-n$ and divide by y^n

$$\frac{du}{dx} + (1-n)\, P(x) \underbrace{\frac{y}{y^n}}_{= y^{1-n} = u} = (1-n)\, f(x)$$

so u solves

$$\frac{du}{dx} + (1-n)\, P(x)\, u = (1-n)\, f(x)$$

we can write the 1st order linear ODE for u

as $$\frac{du}{dx} + P_1(x)\, u = f_1(x)$$

where $P_1(x) = (1-n)P(x)$ and

$$f_1(x) = (1-n)f(x)$$

From $u = y^{1-n}$, $y = u^{\frac{1}{1-n}}$

Solving a Bernoulli Equation $\frac{dy}{dx} + P(x)y = f(x)y^n$

- Introduce the new dependent variable $u = y^{1-n}$.
- Then u solves the first order linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x).$$

- Solve this linear equation using an integrating factor (in the usual way).
- Substitute back to the original variable

$$y = u^{\frac{1}{1-n}}.$$

Example

Solve the initial value problem

$$\frac{dy}{dx} + \frac{2}{x}y = x^3y^3, \quad x > 0, \quad y(1) = \frac{1}{2}.$$

The equation is Bernoulli with

$n=3$, $P(x) = \frac{2}{x}$, $f(x) = x^3$, $1-n = 1-3 = -2$

Set $u = y^{1-n} = y^{-2}$. u solves

$$\frac{du}{dx} + (1-n)P(x)\,u = (1-n)f(x)$$

$$\frac{du}{dx} + (-2)\frac{2}{x}u = (-2)x^3$$

$$\frac{du}{dx} - \frac{4}{x}u = -2x^3$$ 1st order linear

$$P_1(x) = \frac{-4}{x}, \quad \mu = e^{\int P_1(x)\,dx} = e^{\int \frac{-4}{x}dx} = e^{-4\ln x}$$

$$= e^{\ln x^{-4}} = x^{-4}$$

Multiply by μ

$$\frac{d}{dx}\left(x^{-4}u\right) = x^{-4}(-2x^3) = -2x^{-1}$$

$$\int \frac{d}{dx}(x^{-4}u)\,dx = \int \frac{-2}{x}\,dx$$

$$x^{-4}u = -2\ln x + C$$

$$u = \frac{-2\ln x + C}{x^{-4}} = -2x^4\ln x + Cx^4$$

$u = Cx^4 - 2x^4 \ln x$

We need to solve for y and apply $y(1) = \frac{1}{2}$.

We can turn $y(1) = \frac{1}{2}$ into a condition on u.

$$u = y^{-2} \Rightarrow u(1) = (y(1))^{-2} = \left(\tfrac{1}{2}\right)^{-2} = 4$$

$$u(1) = 4 \Rightarrow u(1) = C(1)^4 - 2(1)^4 \ln 1 = C = 4$$

$$C = 4$$

So $u = 4x^4 - 2x^4 \ln x$

Since $u = y^{-2}$, $y = u^{-1/2}$

The solution to the IVP is

$$\boxed{y = \left(4x^4 - 2x^4 \ln x\right)^{-1/2}}$$

Since $y(1) > 0$, $u = y^{-2} \implies y = u^{-1/2}$

and not $y = -u^{-1/2}$