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Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0, \quad \text{with } a\neq 0.$$

If the number m is a solution to the characteristic equation[2]

$$am^2+bm+c=0,$$

then $y=e^{mx}$ is a solution to the differential equation. There are three cases for m.

[1] We'll extend the result to higher order at the end of this section.

[2] The expression am^2+bm+c is the characteristic polynomial, and the equation $am^2+bm+c=0$ is called the characteristic or auxiliary equation.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx}.$$

Example

Solve the IVP

2nd order linear homogeneous and constant coef.

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

The characteristic equation is

$$m^2 + 6m + 9 = 0$$

factor $(m+3)^2 = 0 \implies m = -3$ repeated root

so $y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$

The general solution

$$y = c_1 e^{-3x} + c_2 x e^{-3x}$$

Apply the IC.

$$y' = -3C_1 e^{-3x} + C_2 e^{-3x} - 3C_2 x e^{-3x}$$

$$y(0) = C_1 e^0 + C_2 \cdot 0 \cdot e^0 = 4 \implies C_1 = 4$$

$$y'(0) = -3C_1 e^0 + C_2 e^0 - 3C_2 \cdot 0 \cdot e^0 = 0$$

$$-3C_1 + C_2 = 0 \implies C_2 = 3C_1 = 3 \cdot 4 = 12$$

The solution to the IVP

$$y = 4e^{-3x} + 12x e^{-3x}$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

The two roots of the characteristic equation will be

$$m_1 = \alpha + i\beta \quad \text{and} \quad m_2 = \alpha - i\beta \quad \text{where} \quad i^2 = -1.$$

We want our solutions in the form of *real valued* functions. We start by writing a pair of solutions

$$Y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x}.$$

We will use the **principle of superposition** to write solutions y_1 and y_2 that do not contain the complex number i.

Deriving the solutions Case III

Recall Euler's Formula[3] : $e^{i\theta} = \cos\theta + i\sin\theta$.

$$Y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$Y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

Let $y_1 = \frac{1}{2}Y_1 + \frac{1}{2}Y_2 = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$

Let $y_2 = \frac{1}{2i}Y_1 - \frac{1}{2i}Y_2 = \frac{1}{2i}\left(2i\,e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$

[3] As the sine is an odd function $e^{-i\theta} = \cos\theta - i\sin\theta$.

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x)$$

General solution

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

Let α be the real part of the complex roots and β be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Example

2nd order linear homogeneous constant coef.

Find the general solution of $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0.$

Characteristic eqn

$$m^2 + 4m + 6 = 0$$

Completing the square

$$m^2 + 4m + 4 - 4 + 6 = 0$$

$$(m+2)^2 + 2 = 0$$

$$(m+2)^2 = -2$$

$$m + 2 = \pm\sqrt{-2} = \pm i\sqrt{2}$$

$m = -2 \pm i\sqrt{2}$ $\qquad \alpha \pm i\beta$

Complex case with $\alpha = -2$ and $\beta = \sqrt{2}$

The solutions

$$x_1 = e^{-2t}\cos(\sqrt{2}\,t), \quad x_2 = e^{-2t}\sin(\sqrt{2}\,t)$$

The general solution

$$x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.

- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ for each pair of complex roots.

- It may require a computer algebra system to find the roots for a high degree polynomial.

Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, m may be a root of multiplicity k where $1 \leq k \leq n$.

- If a real root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \dots, \quad x^{k-1} e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \dots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

Example

Find the general solution of the ODE.

$$y''' + y'' + 4y' + 4y = 0$$

3rd order, linear, homogeneous constant coef.

We should have 3 lin. independ. soln.

The characteristic equation is

$$m^3 + m^2 + 4m + 4 = 0$$

factoring by grouping

$$m^2(m+1) + 4(m+1) = 0$$

$$(m+1)(m^2+4) = 0$$

$m+1 = 0 \implies m = -1$ real root

$$m^2+4=0 \Rightarrow m^2=-4 \Rightarrow m=\pm\sqrt{-4}=\pm i2$$

$$m=0\pm i2$$

Complex conjugates with $\alpha=0$, $\beta=2$

From $m=-1$, $y_1=e^{-x}$

$m=0\pm 2i$ $\quad y_2=e^{0x}\cos(2x)$ $\quad y_3=e^{0x}\sin(2x)$

The general solution

$$y=c_1e^{-x}+c_2\cos(2x)+c_3\sin(2x)$$

Example

Find the general solution of the ODE.

$$y''' - 3y'' + 3y' - y = 0$$

3rd order, linear, homogeneous
constant coef.
we need 3 lin. indep. solns

The characteristic eqn is

$$m^3 - 3m^2 + 3m - 1 = 0$$

This is $(m-1)^3 = 0 \Rightarrow m = 1$ is a triple root

The solutions are

$$y_1 = e^{1x} = e^x, \quad y_2 = xe^x, \quad y_3 = x^2 e^x$$

The general solution

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

Example

7th order linear homogeneous constant coeff.

The ODE
$y^{(7)} - 5y^{(6)} + 11y^{(5)} - 31y^{(4)} + 40y^{(3)} - 8y'' + 48y' + 144y = 0$ has characteristic polynomial

$$(m^2+4)^2(m-3)^2(m+1).$$

Determine the general solution.

We need 7 lin. indep. solutions.

$(m^2+4)^2(m-3)^2(m+1) = 0$

$m+1 = 0 \Rightarrow m = -1$ single real root

$y_1 = e^{-x}$

$(m-3)^2 = 0 \Rightarrow m = 3$ double root

$$y_2 = e^{3x}, \quad y_3 = x e^{3x}$$

$$(m^2+4)^2 = 0 \implies m^2 + 4 = 0$$

$$m = \pm i2 = 0 \pm i2$$

$\alpha = 0$, $\beta = 2$ are for double complex conjugate roots

$$y_4 = e^{0x}\cos(2x) \qquad y_5 = e^{0x}\sin(2x)$$

$$y_6 = x e^{0x}\cos(2x) \qquad y_7 = x e^{0x}\sin(2x)$$

Gen. Soln

$$y = c_1 e^{-x} + c_2 e^{3x} + c_3 x e^{3x} + c_4 \cos 2x + c_5 \sin 2x$$
$$+ c_6 x \cos(2x) + c_7 x \sin(2x)$$