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Section 6: Linear Equations Theory and Terminology

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (1)$$

Definition: General Solution of Homogeneous, Linear ODE

Let y_1, $y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation (1). Then the **general solution** of (1) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Remark: We're ready to consider **nonhomogeneous**, linear ODEs. We will use the term general solution slightly differently in the nonhomogeneous context.

Reminder

Last time, we verified that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2 y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and we said that the **general solution** of THIS homogeneous ODE is

$$y = c_1 x^2 + c_2 x^3.$$

Nonhomogeneous Equations

Now we turn our attention to nonhomogeneous equations. We will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (2)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** of (2) is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

This equation has the same left hand side as (2). It's simply the homogeneous version of (2).

General Solution (nonhomogeneous)

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad (2)$$

Definition: General Solution of Nonhomogeneous, Linear ODE

Let y_p be any solution of the nonhomogeneous equation (2), and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the (2) is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) = y_c + y_p$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note that $\boxed{y_c = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}$.

Another Superposition Principle

Consider the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_1(x) + g_2(x) \qquad (3)$$

Theorem: Superposition Principle Nonhomogeneous ODE

Theorem: If y_{p_1} is a particular solution for

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x),$$

and y_{p_2} is a particular solution for

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_2(x),$$

then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution for the nonhomogeneous equation (3).

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

Sub in y_{p_1}

$y_{p_1} = 6$

$y_{p_1}' = 0$

$y_{p_1}'' = 0$

$x^2 y_{p_1}'' - 4x y_{p_1}' + 6 y_{p_1} \stackrel{?}{=} 36$

$x^2(0) - 4x(0) + 6(6) \stackrel{?}{=} 36$

$36 = 36$ ✓

y_{p_1} does solve this ODE.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

Substitute

$y_{p_2} = -7x$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$

$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$

$28x - 42x \stackrel{?}{=} -14x$

$-14x = -14x$ ✓

yes y_{p_2} does solve this ODE

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$y = y_c + y_p$, $y_c = c_1y_1 + c_2y_2 = c_1x^2 + c_2x^3$

By our 2nd principle of superposition

$$y_p = y_{p_1} + y_{p_2} = 6 - 7x$$

The general solution is

$$y = c_1x^2 + c_2x^3 + 6 - 7x$$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = 5$$

The general solution is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

$$y' = 2c_1 x + 3c_2 x^2 - 7$$

$$y(1) = c_1 (1)^2 + c_2 (1)^3 + 6 - 7(1) = 0$$

$$y'(1) = 2c_1(1) + 3c_2(1)^2 - 7 = 5$$

$$c_1 + c_2 - 1 = 0 \implies c_1 + c_2 = 1$$

$$2c_1 + 3c_2 - 7 = 5 \implies 2c_1 + 3c_2 = 12$$

solve this system

$$\begin{aligned} 2c_1 + 2c_2 &= 2 \\ - \quad 2c_1 + 3c_2 &= 12 \\ \hline -c_2 &= -10 \end{aligned} \quad \Rightarrow \quad c_2 = 10$$

$$c_1 = 1 - c_2 = 1 - 10 = -9$$

The solution to the IVP is

$$y = -9x^2 + 10x^3 + 6 - 7x$$

Section 7: Reduction of Order

In sections 7 and 8, we will consider finding solutions to some linear, homogeneous differential equations. In this section, we'll only consider second order homogeneous equations. To motivate the topic:

Consider the second order homogeneous ODE

$$x^2y'' - xy' + y = 0 \quad \text{for} \quad x > 0.$$

- Note that $y_1 = x$ is a solution.
- **Question:** Is $y = c_1y_1$ the general solution? (Why/why not?)

No, it's a 2^{nd} order eqn. There should be y_2

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Standard Form

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Some things to keep in mind:

- Every fundamental solution set has two linearly independent solutions y_1 and y_2,
- The general solution will be

$$y = c_1y_1(x) + c_2y_2(x).$$

Suppose we know one solution $y_1(x)$. This section is about a process called **Reduction of order.** Reduction of order is a method for finding a second solution by assuming that

$$y_2(x) = u(x)y_1(x).$$

The goal is to find the unknown function u.

Context

- We start with a second order, linear, homogeneous ODE in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

- We know one solution y_1. (Keep in mind that y_1 is a known!)
- We know there is a second linearly independent solution (section 6 theory says so).
- We try to find y_2 by guessing that it can be found in the form

$$y_2(x) = u(x)y_1(x)$$

where the goal becomes finding u.

- **Due to linear independence, we know that u cannot be constant.**

Example

Find the general solution to the ODE $x^2y'' - xy' + y = 0$ for $x > 0$ given that $y_1(x) = x$ is one solution.

y₂ is supposed to solve the ODE, so we plug it in.

Suppose $y_2 = u(x)y_1(x)$

$y_2 = xu$ sub into the ODE

The eqn in standard form is

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$$

$$y_2 = xu$$

$$y_2' = xu' + 1u = xu' + u$$

$$y_2'' = xu'' + 1u' + u' = xu'' + 2u'$$

$$y_2'' - \frac{1}{x} y_2' + \frac{1}{x^2} y_2 = 0$$

$$x u'' + 2u' - \frac{1}{x}(x u' + u) + \frac{1}{x^2}(x u) = 0$$

$$x u'' + 2u' - u' - \frac{1}{x} u + \frac{1}{x} u = 0$$

$$x u'' + u' = 0$$

let $w = u'$, $w' = u''$

1st order

$$x w' + w = 0$$

separate

$$x w' = -w$$

$$\frac{1}{w} \frac{dw}{dx} = \frac{-1}{x}$$

$$\int \frac{1}{w}\, dw = \int \frac{-1}{x}\, dx$$

$$\ln|w| = -\ln|x| + C$$

$$e^{\ln|w|} = e^{-\ln|x| + C} = e^{\ln x^{-1}} \cdot e^{C}$$

$$|w| = e^{c} x^{-1} \qquad \pm e^{c} = k$$

$$w = k x^{-1}$$

$$u' = w \implies u' = \frac{k}{x}$$

$$u = \int \frac{k}{x}\, dx = k \ln x$$

$y_2 = u y_1 = (k \ln x) x = k x \ln x$

The general solution

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 x + c_2 x \ln x$$

Note: The k factor was dropped because we're including the coefficient c_2.