September 20 Math 2306 sec. 51 Spring 2023

Section 6: Linear Equations Theory and Terminology

We're considering n^{th} order, linear equations

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (1)$$

and assuming that $a_0, \ldots, a_n$ and g are continuous on some interval I and $a_n(x) \neq 0$ on I.

The goal is to determine what the *general solution* of (1) should look like. For now, we are focused only on the homogeneous case (right side is zero)

$$\boxed{a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.}$$

Recall Superposition

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Theorem: The Principle of Superposition

If y_1, $y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This says that we can build new solutions from existing ones by (1) multiplying by constants and (2) adding.

Linear Dependence

Definition: Linear Dependence

A set of functions $f_1(x)$, $f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants c_1, $c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

Definition: Linear Independence

A set of functions $f_1(x)$, $f_2(x), \ldots, f_n(x)$ are said to be **linearly independent** on an interval I if the equation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I$$

is only true when $c_1 = c_2 = \cdots = c_n = 0$.

Example of Linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are **linearly independent** on $(-\infty, \infty)$ because

$$c_1 \sin(x) + c_2 \cos(x) = 0 \quad \text{for all real } x$$

is ONLY true if $c_1 = 0$ and $c_2 = 0$.

Example of Linearly Dependent Set

In contrast, the set of functions $f_1(x) = x^2$, $f_2(x) = 4x$, and $f_3(x) = x - x^2$ is **linearly dependent** on $(-\infty, \infty)$ because for every real x

$$4f_1(x) - f_2(x) + 4f_3(x) = 0. \tag{2}$$

We called the equation (2) a **linear dependence relation.**

Definition: Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

In the case of two functions, $\{y_1, y_2\}$, the Wronskian is

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Determinant Formulas (2×2 and 3×3)

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

The matrix will be 3x3.

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1-2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2(-8) - 4x\left(-4x - 2(1-2x)\right) + (x-x^2)(-8)$$

$$-4x - 2 + 4x$$

$$= -8x^2 - 4x(-2) - 8x + 8x^2$$

$$= -8x^2 + 8x - 8x + 8x^2$$

$$= 0$$

$$W(f_1, f_2, f_3)(x) = 0$$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

Theorem (a test for linear independence)

Let f_1, $f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that

$$W(f_1, f_2, \ldots, f_n)(x_0) \neq 0,$$

then the functions are **linearly independent** on I.

We can use this as a test for linear dependence.

$$W \neq 0 \quad \Longrightarrow \quad \text{Independent}$$

It is possible to construct a pair of linearly independent functions whose Wronskian is zero. But such a pair can't be solutions to the same linear ODE. So for our purposes, we can say $W = 0$ implies linear dependence.

Example

Determine whether the functions are linearly dependent or linearly independent on the given interval.

$$y_1 = x^2, \quad y_2 = x^3 \quad I = (0, \infty)$$

We can use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= x^2(3x^2) - 2x(x^3)$$

$$= 3x^4 - 2x^4$$

$$= x^4$$

Since $W(y_1, y_2)(x) \neq 0$, the

functions are linearly

independent.

Fundamental Solution Set

We continue to consider the n^{th} order, linear, homogeneous ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

We're ready to get at what the solution to a homogeneous linear ODE will be. First, a definition.

Definition: Fundamental Solution Set

A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Fundamental Solution Set

Theorem

If $a_1, a_2, \dots, a_n$ are continuous on an interval I and $a_n(x) \neq 0$ for every x in I, then the homogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

possess a fundamental solution set.

So under the conditions on the coefficients that we've stated, a fundamental solution exists. The next definition tells us what the general solution to the ODE is.

General Solution of n^{th} order Linear Homogeneous Equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (3)$$

Definition: General Solution of Homogeneous, Linear ODE

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation (3). Then the **general solution** of (3) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Remark: This indicates that the task of solving an n^{th} order linear **homogeneous** ODE is to find a fundamental solution set, i.e., n, linearly independent solutions. We build the general solution by creating a linear combination.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

Since the ODE is 2nd order, we have to show that we have two, linearly independent solutions.

- Since $W(y_1, y_2)(x) = x^4 \neq 0$, this is a linearly independent pair of functions.

Let's show that they are solutions

$$x^2 y'' - 4xy' + 6y = 0$$

$y_1 = x^2$ and $y_2 = x^3$

$y_1' = 2x$ $\quad$ $y_2' = 3x^2$

$y_1'' = 2$ $\quad$ $y_2'' = 6x$

y_1: $\quad x^2 y_1'' - 4x y_1' + 6y_1 \stackrel{?}{=} 0$

$$x^2(2) - 4x(2x) + 6(x^2) \stackrel{?}{=} 0$$

$$2x^2 - 8x^2 + 6x^2 \stackrel{?}{=} 0$$

$$0 = 0 \quad \checkmark$$

y_1 is a solution

y_2: $\quad x^2 y_2'' - 4x y_2' + 6y_2 \stackrel{?}{=} 0$

$$x^2(6x) - 4x(3x^2) + 6(x^3) \stackrel{?}{=} 0$$

$$6x^3 - 12x^3 + 6x^3 \stackrel{?}{=} 0$$

y_2 is a solution

$0 \overset{\checkmark}{=} 0$

We've shown that these two functions are solutions and are linearly independent. Hence, they form a fundamental solution set.

The general solution is

$$y = c_1 y_1 + c_2 y_2,$$

i.e., $y = c_1 x^2 + c_2 x^3$