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Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0, \quad \text{with } a\neq 0.$$

If the number m is a solution to the characteristic equation[2]

$$am^2+bm+c=0,$$

then $y=e^{mx}$ is a solution to the differential equation. There are three cases for m.

[1] We'll extend the result to higher order at the end of this section.

[2] The expression am^2+bm+c is the characteristic polynomial, and the equation $am^2+bm+c=0$ is called the characteristic or auxiliary equation.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx}.$$

Example

Solve the IVP

2nd order, linear, homogeneous constant coef.

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

The characteristic equation is

$$m^2 + 6m + 9 = 0$$

This factors as $(m+3)^2 = 0 \Rightarrow m = -3$ double root

Hence $y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$.

The general solution

$$y = c_1 e^{-3x} + c_2 x e^{-3x}$$

We'll apply the I.C.

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$$

$$y(0) = 4 = c_1 e^0 + c_2 \cdot 0 \cdot e^0 \implies c_1 = 4$$

$$y'(0) = 0 = -3c_1 e^0 + c_2 e^0 - 3c_2 \cdot 0 \cdot e^0$$

$$-3c_1 + c_2 = 0 \implies c_2 = 3c_1 = 3(4) = 12$$

The solution to the IVP is

$$\boxed{y = 4e^{-3x} + 12x e^{-3x}.}$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

The two roots of the characteristic equation will be

$$m_1 = \alpha + i\beta \quad \text{and} \quad m_2 = \alpha - i\beta \quad \text{where} \quad i^2 = -1.$$

We want our solutions in the form of *real valued* functions. We start by writing a pair of solutions

$$Y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x}.$$

We will use the **principle of superposition** to write solutions y_1 and y_2 that do not contain the complex number i.

Deriving the solutions Case III

Recall Euler's Formula[3] : $e^{i\theta} = \cos\theta + i\sin\theta$.

$$Y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$Y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

Let $y_1 = \frac{1}{2}Y_1 + \frac{1}{2}Y_2 = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$

Let $y_2 = \frac{1}{2i}Y_1 - \frac{1}{2i}Y_2 = \frac{1}{2i}\left(2ie^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$

[3] As the sine is an odd function $e^{-i\theta} = \cos\theta - i\sin\theta$.

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

Let α be the real part of the complex roots and β be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Example

2nd order, linear, homogeneous, constant coef.

Find the general solution of $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 6x = 0.$

The characteristic equation is

$$m^2 + 4m + 6 = 0$$

Using the quadratic formula

$$m = \frac{-4 \pm \sqrt{4^2 - 4\cdot1\cdot6}}{2\cdot1} = \frac{-4 \pm \sqrt{16-24}}{2} = \frac{-4 \pm \sqrt{-8}}{2}$$

$$= \frac{-4 \pm i\sqrt{8}}{2} = \frac{-4 \pm i2\sqrt{2}}{2} = -2 \pm i\sqrt{2}$$

$m = \alpha \pm i\beta$ $\qquad$ $\alpha = -2$ and $\beta = \sqrt{2}$

So $x_1 = e^{-2t}\cos(\sqrt{2}\,t)$ and $x_2 = e^{-2t}\sin(\sqrt{2}\,t)$

The general solution

$$x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.

- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ for each pair of complex roots.

- It may require a computer algebra system to find the roots for a high degree polynomial.

Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, m may be a root of multiplicity k where $1 \leq k \leq n$.

- If a real root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \dots, \quad x^{k-1} e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \dots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

Example

Find the general solution of the ODE.

$$y''' + y'' + 4y' + 4y = 0$$

3rd order, linear, homogeneous, constant coefficient

we need 3 lin. independent solns.

The characteristic eqn is

$$m^3 + m^2 + 4m + 4 = 0$$

factor by grouping

$$m^2(m+1) + 4(m+1) = 0$$

$$(m+1)(m^2+4) = 0$$

$m+1=0 \Rightarrow m=-1$ real nonrepeated root. e^{-1x}

$m^2+4=0 \Rightarrow m^2=-4 \Rightarrow m=\pm\sqrt{-4}=\pm i2$

$m=\alpha\pm i\beta$ $\quad m=0\pm i2 \quad \alpha=0,\ \beta=2$

From $m=-1$, $y_1=e^{-x}$

From $m=0\pm i2$, $y_2=e^{0x}\cos(2x)=\cos(2x)$

$y_3=e^{0x}\sin(2x)=\sin(2x)$

The general solution

$$y=c_1 e^{-x}+c_2\cos(2x)+c_3\sin(2x)$$

Example

Find the general solution of the ODE.

$y''' - 3y'' + 3y' - y = 0$

3^{rd} order, linear, homogeneous

Constant coef.

Characteristic eqn.

$$m^3 - 3m^2 + 3m - 1 = 0$$

This is $(m-1)^3 = 0 \Rightarrow m = 1$ triple root.

Hence $y_1 = e^x$, $y_2 = xe^x$, $y_3 = x^2 e^x$

The general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

Example

7th order, linear, homogeneous, constant coef.

The ODE
$y^{(7)} - 5y^{(6)} + 11y^{(5)} - 31y^{(4)} + 40y^{(3)} - 8y'' + 48y' + 144y = 0$ has characteristic polynomial

$$(m^2+4)^2(m-3)^2(m+1).$$

Determine the general solution.

There are 7 lin. independent functions

$$(m^2+4)^2(m-3)^2(m+1) = 0$$

From $m+1=0 \implies m=-1$ real nonrepeated

$$y_1 = e^{-x}$$

From $(m-3)^2=0 \implies m=3$ double real root

$y_2 = e^{3x}$, $y_3 = xe^{3x}$

From $(m^2+4)^2 = 0 \Rightarrow m^2+4=0$

$\Rightarrow m = \pm i2$

each is a double root.

$y_4 = e^{0x}\cos(2x)$, $y_5 = e^{0x}\sin(2x)$

$y_6 = xe^{0x}\cos(2x)$, $y_7 = xe^{0x}\sin(2x)$

The general solution is

$$y = c_1 e^{-x} + c_2 e^{3x} + c_3 x e^{3x} + c_4 \cos(2x) + c_5 \sin(2x) + \\ + c_6 x \cos(2x) + c_7 x \sin(2x)$$