September 20 Math 2306 sec. 54 Fall 2021

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order¹, linear, homogeneous equation with constant coefficients

$$arac{d^2y}{dx^2}+brac{dy}{dx}+cy=0, \quad ext{with } a
eq 0.$$

If the number m is a solution to the characteristic equation²

$$am^2+bm+c=0,$$

then $y = e^{mx}$ is a solution to the differential equation. There are three cases for *m*.

¹We'll extend the result to higher order at the end of this section.

²The expression $am^2 + bm + c$ is the characteristic polynomial, and the equation $am^2 + bm + c = 0$ is called the characteristic or auxiliary equation.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac > 0$.

There are two different roots m_1 and m_2 . A fundamental solution set consists of

$$y_1 = e^{m_1 x}$$
 and $y_2 = e^{m_2 x}$.

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

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Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac = 0$

If the characteristic equation has one real repeated root *m*, then a fundamental solution set to the second order equation consists of

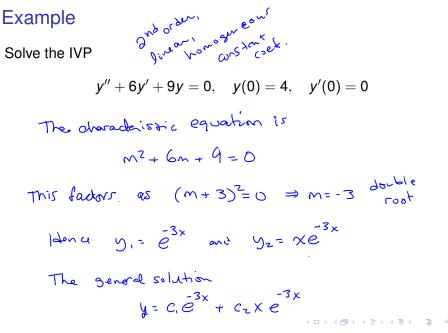
$$y_1 = e^{mx}$$
 and $y_2 = xe^{mx}$.

The general solution is

$$y=c_1e^{mx}+c_2xe^{mx}.$$

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Well apply the I.C $y' = -3c_1e^{3x} + c_2e^{-3x} - 3c_2 \times e^{-3x}$ $y(0) = 4 = C_1 e^{\theta} + C_2 \cdot 0 \cdot e^{\theta} \implies C_1 = 4$ $y'(0)=0 = -3c, e' + c_2 e' - 3c_2 \cdot 0 \cdot e'$ $-3c_1+c_2 = 0 \implies c_2 = 3c_1 = 3(4) = 12.$ he solution to the IVP' is $y = 4e^{-3x} + 12xe^{-3x}$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac < 0$

The two roots of the characteristic equation will be

$$m_1 = \alpha + i\beta$$
 and $m_2 = \alpha - i\beta$ where $i^2 = -1$.

We want our solutions in the form of <u>real valued</u> functions. We start by writing a pair of solutions

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}$$
, and $Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}$.

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We will use the **principle of superposition** to write solutions y_1 and y_2 that do not contain the complex number *i*.

Deriving the solutions Case III

Recall Euler's Formula³ : $e^{i\theta} = \cos \theta + i \sin \theta$.

$$Y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x} \left(Cos(\beta x) + i Sin(\beta x) \right)$$

$$Y_{2} = e^{\alpha x} e^{-i\beta x} = e^{\alpha x} \left(Cos(\beta x) - i Sm(\beta x) \right)$$

Let $y_{1} = \frac{1}{2} Y_{1} + \frac{1}{2} Y_{2} = \frac{1}{2} \left(2e^{\alpha x} Cos(\beta x) \right) = e^{\alpha x} Cos(\beta x)$
Let $y_{2} = \frac{1}{2i} Y_{1} - \frac{1}{2i} Y_{2} = \frac{1}{2i} \left(2ie^{\alpha x} Sm(\beta x) \right) = e^{\alpha x} Sin(\beta x)$

³As the sine is an odd function $e^{-i\theta} = \cos \theta - i \sin \theta$.

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac < 0$

Let α be the real part of the complex roots and β be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x} \cos(\beta x)$$
 and $y_2 = e^{\alpha x} \sin(\beta x)$.

The general solution is

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x).$$

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Find the general solution of

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+6x=0.$$

.

The characteristic equation is

$$m^{2} + 4m + 6 = 0$$
Using the quadratic formula

$$M = -\frac{4 \pm \sqrt{2^{2} - 4 \cdot 1 \cdot 6}}{2 \cdot 1} = -\frac{4 \pm \sqrt{16 - 24}}{2} = -\frac{4 \pm \sqrt{18}}{2}$$

$$= -\frac{4 \pm \sqrt{18}}{2} = -\frac{4 \pm \sqrt{16} - 24}{2} = -2 \pm \sqrt{12}$$

$$M = 9 \pm \sqrt{16}$$

$$Q = -2 \text{ and } \beta = \sqrt{12}$$

So $X_1 = e^{2t} \cos(\sqrt{2t}) = x_2 = e^{2t} \sin(\sqrt{2t})$

The general solution $X = C_i e^{-2t} cor(\sqrt{2}t) + (2e^{-2t}sin(\sqrt{2}t))$

Higer Order Linear Constant Coefficient ODEs

The same approach applies. For an nth order equation, we obtain an nth degree polynomial.

Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions e^{αx} cos(βx) and e^{αx} sin(βx) for each pair of complex roots.

It may require a computer algebra system to find the roots for a high degree polynomial.

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Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, *m* may be a root of multiplicity *k* where $1 \le k \le n$.
- If a real root m is repeated k times, we get k linearly independent solutions

$$e^{mx}$$
, xe^{mx} , x^2e^{mx} , ..., $x^{k-1}e^{mx}$

or in conjugate pairs cases 2k solutions

$$e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x), xe^{\alpha x}\cos(\beta x), xe^{\alpha x}\sin(\beta x), \dots,$$

 $x^{k-1}e^{\alpha x}\cos(\beta x), x^{k-1}e^{\alpha x}\sin(\beta x)$

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Find the general solution of the ODE.

$$y'''+y''+4y'+4y=0$$

 3^{r^2} order linear, homogeneous,
constant coefficient
we need 3 lin. independent
solve.

The characteristic eqn is

$$M^{3} + M^{2} + YM + Y = 0$$
factor by grouping

$$M^{2}(m+1) + Y(m+1) = 0$$

$$(m+1)(m^{2} + Y) = 0$$
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 $m+1=0 \implies m=-1 \text{ real non repeated} e^{1x}$ $m^{2}+Y=0 \implies m^{2}=-Y \implies m=\pm \int -Y=\pm iZ$ $m=d\pm i\beta \qquad m=0\pm iZ \qquad q=0, \beta=Z$

From
$$m=-1$$
, $y_1 = e^{x}$
From $m=0\pm iz$, $y_2 = e^{x}Cas(z_x) = Cas(z_x)$
 $y_3 = e^{x}Sin(z_x) = Sin(z_x)$

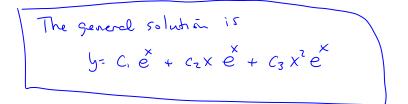
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Find the general solution of the ODE.

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Observations the eqn. $M^3 - 3M^2 + 3M - 1 = 0$ This is $(M-1)^3 = 0 \implies M = 1$ the plane Hence $y_1 = e^{X}$, $y_2 = xe^{X}$, $y_3 = x^2e^{X}$

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The ODE $y^{(7)} - 5y^{(6)} + 11y^{(5)} - 31y^{(4)} + 40y^{(3)} - 8y'' + 48y' + 144y = 0$ has characteristic polynomial

$$(m^2+4)^2(m-3)^2(m+1).$$

Determine the general solution.

There are 7 lin. independent functions

$$(m^2+y)^2(m-3)^2(m+1) = 0$$

From $m+1=0 \implies m=-1$ real repeated
 $y_1 = e^{x}$
From $(m-3)^2=0 \implies m=3$ dowble real
 $(m^2+y)^2(m-3)^2 = 0 \implies m=3$ dowble real

$$y_{2} = e^{3x} , y_{3} = xe^{3x}$$
From $(m^{2}+y)^{2}=0 \Rightarrow m^{2}+y=0$
 $\Rightarrow m = \pm i2$ e^{acha}
 $y_{4} = e^{x}\cos(2x) , y_{5} = e^{x}\sin(2x)$
 $y_{6} = xe^{x}\cos(2x) , y_{7} = xe^{x}\sin(2x)$
The general solution is
 $y_{7} = C_{1}e^{x} + c_{2}e^{3x} + c_{3}xe^{3x} + c_{4}Gs(2x) + c_{5}Sm(2x) + t_{7}e^{5x}\cos(2x) + t_{7}e^{5$