September 21 Math 2306 sec. 52 Fall 2022

Section 7: Reduction of Order

- We start with a second order, linear, homogeneous ODE in standard form
$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0.$$
- We know one solution y_1. (Keep in mind that y_1 is a known!)
- We know there is a second linearly independent solution (section 6 theory says so).
- We try to find y_2 by guessing that it can be found in the form
$$y_2(x)=u(x)y_1(x)$$
where the goal becomes finding u.
- **Due to linear independence, we know that u cannot be constant.**

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - \text{is known.}$$

Suppose

$$y_2 = uy_1$$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

We know that $\cdot y_1'' + P(x)y_1' + Q(x)y_1 = 0.$

Sub into the ODE

$$y_2'' + P(x)\,y_2' + Q(x)\,y_2 = 0$$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)\left(u'y_1 + uy_1'\right) + Q(x)\,uy_1 = 0$$

Collect u'', u', and u

$$y_1u'' + \left(2y_1' + P(x)y_1\right)u' + \underbrace{\left(y_1'' + P(x)y_1' + Q(x)y_1\right)}_{\text{"0" because } y_1 \text{ solves the ODE}}u = 0$$

The equation for u is

$$y_1u'' + \left(2y_1' + P(x)y_1\right)u' = 0$$

Let $w = u'$, then $w' = u''$ and w solves

$$y_1 w' + (2y_1' + P(x) y_1) w = 0$$

Assume $y_1(x) \neq 0$ on the domain and $w > 0$

$$\frac{dw}{dx} + 2\left(\frac{\frac{dy_1}{dx}}{y_1} + P(x)\right) w = 0$$

This is 1^{st} order linear and separable.

Let's separate variables

$$\frac{dw}{dx} = -\left(2\,\frac{\frac{dy_1}{dx}}{y_1} + P(x)\right) w$$

$$\frac{1}{w}\frac{dw}{dx} = -\left(2\frac{\frac{dy_1}{dx}}{y_1} + P(x)\right)$$

$$\frac{1}{w}\frac{dw}{dx}\,dx = -\left(2\frac{\frac{dy_1}{dx}}{y_1} + P(x)\right)dx$$

$$\frac{1}{w}\frac{dw}{dx}\,dx = -2\frac{\frac{dy_1}{dx}}{y_1}\,dx - P(x)\,dx$$

$$\frac{1}{w}\,dw = -2\frac{dy_1}{y_1} - P(x)\,dx$$

$$\int\frac{1}{w}\,dw = -2\int\frac{dy_1}{y_1} - \int P(x)\,dx$$

$$\ln w = -2\ln|y_1| - \int P(x)\,dx$$

$$\ln w = \ln y_1^{-2} - \int P(x)\,dx$$

Exponentiate

$$w = e^{\ln y_1^{-2} - \int P(x)\,dx}$$

$$= e^{\ln y_1^{-2}} \cdot e^{-\int P(x)\,dx}$$

$$w = \frac{1}{y_1^2}\, e^{-\int P(x)\,dx}$$

$$w = u' \implies u = \int w\,dx$$

$$\Rightarrow \quad u = \int \frac{e^{-\int P(x)dx}}{y_1^2}\, dx$$

$y_2 = u y_1$ and the general solution

$$y = c_1 y_1 + c_2 y_2$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 1$$

Find the general solution $y = c_1 y_1 + c_2 y_2$

We'll use reduction of order to find y_2.

$y_2 = u y_1$ where $u = \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$

The ODE is in standard form, $P(x) = 4$

$$-\int p(x)\,dx = -\int 4\,dx = -4x$$

$$e^{-\int p(x)\,dx} = e^{-4x}, \quad y_1 = e^{-2x} \Rightarrow y_1^2 = \left(e^{-2x}\right)^2 = e^{-4x}$$

$$u = \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx = \int \frac{e^{-4x}}{e^{-4x}}\,dx =$$

$$= \int dx = x$$

$$y_2 = u y_1 = x e^{-2x}, \quad y_1 = e^{-2x}$$

The general solution to the ODE is

$$y = C_1 e^{-2x} + C_2 x e^{-2x}$$

Now apply $y(0)=1$ and $y'(0)=1$

$$y' = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x}$$

$$y(0) = C_1 e^{0} + C_2 \cdot 0 \cdot e^{0} = 1 \implies C_1 = 1$$

$$y'(0) = -2C_1 e^{0} + C_2 e^{0} - 2C_2 \cdot 0 \cdot e^{0} = 1$$

$$-2C_1 + C_2 = 1 \implies C_2 = 1 + 2C_1 = 3$$

The solution to the IVP is

$$y = e^{-2x} + 3x\,e^{-2x}$$