September 22 Math 2306 sec. 51 Fall 2021

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e^{mx}
- sines and/or cosines, $\sin(kx)$ or $\cos(kx)$
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example[1]

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

- left is constant coef.
- right is a polynomial

To find y_c, solve $y'' - 4y' + 4y = 0$

Note that $g(x) = 8x+1$ is a 1st degree polynomial.
We might guess that y_p is also a 1st degree
polynomial. Assume

$y_p = Ax + B$ for constants A and B.

[1] **We're only ignoring the y_c part to illustrate the process.**

Let's sub $y_p = Ax+B$ into the ODE.

$$y_p'' - 4y_p' + 4y_p = 8x+1$$

$y_p = Ax+B$

$y_p' = A$

$y_p'' = 0$

$$0 - 4(A) + 4(Ax+B) = 8x+1$$

$$\Rightarrow \quad 4Ax + (-4A+4B) = 8x+1$$

These polynomials are equal if the powers of x have the same coefficients.

$$\underline{\underline{4A}}x + (\underline{\underline{-4A+4B}}) = \underline{\underline{8}}x + \underline{\underline{1}}$$

Matching sides

$$4A = 8$$
$$-4A+4B = 1$$

$$4A = 8 \Rightarrow A = 2$$

$$-4A + 4B = 1 \Rightarrow 4B = 1 + 4A \Rightarrow B = \frac{1}{4} + A = \frac{1}{4} + 2 = \frac{9}{4}$$

We have our particular solution

$$y_p = 2x + \frac{9}{4}$$

* The general solution would be $y_c + y_p$, so we'd have to find y_c to solve the ODE.

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

Here $g(x) = 6e^{-3x}$. This is a constant times e^{-3x}.

We'll assume $y_p = Ae^{-3x}$.

Sub it in

$$y_p = Ae^{-3x}$$
$$y_p' = -3Ae^{-3x}$$
$$y_p'' = 9Ae^{-3x}$$

$$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$$

$$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4Ae^{-3x} = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x}$$

Matching gives $25A = 6$

$$A = \frac{6}{25}$$

The particular solution

$$y_p = \frac{6}{25}e^{-3x}$$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

The left is constant coef, and the right is a polynomial. $g(x) = 16x^2$.

This can be viewed as a monomial or more generally as a 2nd degree polynomial.

Let's assume $y_p = Ax^2$ (* this turns out to be wrong)

Sub this in $y_p = Ax^2$

$y_p' = 2Ax$

$y_p'' = 2A$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4Ax^2 = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

We get
$$\left.\begin{aligned} 4A &= 16 \\ -8A &= 0 \\ 2A &= 0 \end{aligned}\right\} \Rightarrow$$ inconsistent since $4 \neq 0$.

It can't be true that $y_p = Ax^2$.

To fix this, we need to view $g(x) = 16x^2$ as a 2^{nd} degree polynomial.

Set $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$\underline{\underline{4A}}x^2 + \underline{\underline{(-8A + 4B)}}x + \underline{\underline{(2A - 4B + 4C)}} = \underline{\underline{16}}x^2 + \underline{\underline{0}}x + \underline{\underline{0}}$$

Matching gives

$$\begin{aligned} 4A &= 16 \\ -8A + 4B &= 0 \\ 2A - 4B + 4C &= 0 \end{aligned}$$

$$A = 4, \qquad 4B = 8A \Rightarrow B = 2A = 8$$

$$4C = 4B - 2A \Rightarrow C = B - \frac{1}{2}A = 8 - \frac{1}{2}(4) = 6$$

$$\boxed{\text{We found } y_p = 4x^2 + 8x + 6}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any constant)

Zero degree polynomial

$y_p = A$

(b) $g(x) = x - 7$

1st degree poly.

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ 2nd degree poly.

$y_p = Ax^2 + Bx + C$

(d) $g(x) = 3x^3 - 5$ 3rd degree poly.

$y_p = Ax^3 + Bx^2 + Cx + D$

We'll look at more examples later.