September 22 Math 2306 sec. 51 Spring 2023

Section 6: Linear Equations Theory and Terminology

We continue to consider the n^{th} order, linear, homogeneous ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Definition: Fundamental Solution Set

A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

General Solution of n^{th} order Linear Homogeneous Equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{1}$$

Assume that $a_0, \cdots, a_n$ are continuous on some interval I and that $a_n(x) \neq 0$ for x in I.

Definition: General Solution of Homogeneous, Linear ODE

Let y_1, $y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation (1). Then the **general solution** of (1) is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x),$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Nonhomogeneous Equations

Now we turn our attention to nonhomogeneous equations. We will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (2)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** of (2) is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

This equation has the same left hand side as (2). It's simply the homogeneous version of (2).

Definition: General Solution of Nonhomogeneous, Linear ODE

Let y_p be any solution of the nonhomogeneous equation (2), and let $y_1,\ y_2, \dots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the (2) is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where $c_1,\ c_2, \dots, c_n$ are arbitrary constants.

Note the format

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{y_c} + \underbrace{y_p(x)}_{y_p}$$

Another Superposition Principle

Consider the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_1(x) + g_2(x) \qquad (3)$$

Theorem: Superposition Principle Nonhomogeneous ODE

Theorem: If y_{p_1} is a particular solution for

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x),$$

and y_{p_2} is a particular solution for

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_2(x),$$

then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution for the nonhomogeneous equation (3).

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.

$I = (0, \infty)$

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1}{}' = 0$

$y_{p_1}{}'' = 0$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6y_{p_1} \stackrel{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \stackrel{?}{=} 36$$

$$36 \stackrel{\checkmark}{=} 36$$

So y_{p_1} solves

$$x^2y'' - 4xy' + 6y = 36$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$

$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$

$28x - 42x \stackrel{?}{=} -14x$

$-14x \stackrel{\checkmark}{=} -14x$

So $y_{p_2} = -7x$ solves $x^2y'' - 4xy' + 6y = -14x$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$y = y_c + y_p$ and $y_c = c_1y_1 + c_2y_2$

$= c_1x^2 + c_2x^3$

$y_p = y_{p_1} + y_{p_2}$ $y_p = 6 - 7x$

The general solution is

$$\boxed{y = c_1x^2 + c_2x^3 + 6 - 7x}$$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = 5$$

The general solution is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

Apply the I. C.

$$y' = 2c_1 x + 3c_2 x^2 - 7$$

$$y(1) = c_1(1)^2 + c_2(1)^3 + 6 - 7(1) = 0 \quad \Rightarrow \quad c_1 + c_2 = 1$$

$$y'(1) = 2c_1(1) + 3c_2(1)^2 - 7 = 5 \quad \Rightarrow \quad 2c_1 + 3c_2 = 12$$

Solve the system $c_1 + c_2 = 1$

$2c_1 + 3c_2 = 12$

2 times

$2c_1 + 2c_2 = 2$

subtract

$c_2 = 10$ $\qquad c_1 = 1 - c_2 = 1 - 10 = -9$

$* \quad 2(-9) + 3(10) \stackrel{?}{=} 12 \qquad -18 + 30 = 12$ ✓

The solution to the Initial Value problem is

$$y = -9x^2 + 10x^3 + 6 - 7x$$