September 22 Math 2306 sec. 52 Fall 2021

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e^{mx}
- sines and/or cosines, $\sin(kx)$ or $\cos(kx)$ k, m constants
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example[1]

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

- left side is constant coefficient
- right side is a polynomial
- The general solution would be $y_c + y_p$, but we'll focus on y_p for now.

y_c would solve $y'' - 4y' + 4y = 0$

To find y_p, notice that $g(x) = 8x + 1$ is a 1st degree polynomial. We'll guess that y_p is also a 1st degree polynomial.

[1] **We're only ignoring the y_c part to illustrate the process.**

Suppose $y_p = Ax + B$ with A and B constants. Let's sub it into the ODE $y'' - 4y' + 4y = 8x + 1$.

$$y_p = Ax + B$$
$$y_p' = A$$
$$y_p'' = 0$$

$$y_p'' - 4y_p' + 4y_p = 8x + 1$$

$$0 - 4(A) + 4(Ax + B) = 8x + 1$$

$$\underline{\underline{4A}}x + (\underline{-4A + 4B}) = \underline{\underline{8}}x + \underline{\underline{1}}$$

This will be true if the coefficients of like terms match.

Matching gives

$$4A = 8$$
$$-4A + 4B = 1$$

Solving : $A = \frac{8}{4} = 2$

$4B = 1 + 4A \implies B = \frac{1}{4} + A = \frac{1}{4} + 2 = \frac{9}{4}$

We found the particular solution

$$\boxed{y_p = 2x + \frac{9}{4}}$$

The general solution would be $y_c + y_p$

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

The left is constant coef. The right is exponential.

$g(x) = 6e^{-3x}$ a constant times e^{-3x}.

Guess $y_p = Ae^{-3x}$. Sub into the ODE

$y_p' = -3Ae^{-3x}$

$y_p'' = 9Ae^{-3x}$

$$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$$

$$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4Ae^{-3x} = 6e^{-3x}$$

$$25A\, e^{-3x} = 6\, e^{-3x}$$

This requires $25A = 6 \Rightarrow A = \dfrac{6}{25}$.

$$\boxed{\text{so } \ y_p = \frac{6}{25} e^{-3x}}$$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

The left is constant coef. and the right is a polynomial.

$$g(x) = 16x^2$$

this is a monomial and a polynomial.

Suppose we assume $y_p = Ax^2$ (this will not be correct).

Substitute: $y_p = Ax^2$

$$y_p' = 2Ax$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4Ax^2 = 16x^2$$

zero

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

Matching gives

$$4A = 16$$
$$-8A = 0$$
$$2A = 0$$

not solvable since $4 \neq 0$

The way to think of $g(x) = 16x^2$ is as a 2^{nd} degree polynomial.

Guess $y_p = Ax^2 + Bx + C$

Try again $y_p' = 2Ax + B$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

Matching gives

$$\begin{aligned} 4A &= 16 \\ -8A + 4B &= 0 \\ 2A - 4B + 4C &= 0 \end{aligned}$$

Solving: $4A = 16 \Rightarrow A = 4$

$4B = 8A \Rightarrow B = 2A = 8$

$$4C = 4B - 2A \Rightarrow C = B - \frac{1}{2}A = 8 - \frac{1}{2}(4) = 6$$

The particular solution

$$\boxed{y_p = 4x^2 + 8x + 6}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any constant)

Constant a.k.a. degree zero polynomial

$$y_p = A$$

(b) $g(x) = x - 7$

1st degree polynomial

$$y_p = Ax + B$$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$

2nd degree polynomial

$y_p = Ax^2 + Bx + C$

(d) $g(x) = 3x^3 - 5$

3rd degree polynomial

$y_p = Ax^3 + Bx^2 + Cx + D$