# September 22 Math 2306 sec. 54 Fall 2021

#### Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- ► exponentials, e<sup>mx</sup> ~ ...............
- ► sines and/or cosines, sin(kx), Cos(kx)
- and products and sums of the above kinds of functions

Recall  $y = y_c + y_p$ , so we'll have to find both the complementary and the particular solutions!

k- constant

# Motivating Example<sup>1</sup>

Find a particular solution of the ODE

 We sub this into  $y_p'' - 4y_p' + 4y_p = 8x+1$   $y_p = Ax+B$   $y_p' = A$   $y_p'' = 0$ 4Ax + (-4A+4B) = 8x+1

be con match coefficients  $\frac{YA \times + (-YA + YB)}{This requires} = \frac{8 \times +1}{YA} = 8$ -YA + YB = 1

Solving A=Z

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$$YB = 1 + YA \implies B = \frac{1}{4} + A = \frac{1}{4} + 2 = \frac{9}{4}$$

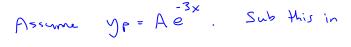
The particular solution is  

$$y_p = Zx + \frac{9}{9}$$

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$$y'' - 4y' + 4y = 6e^{-3x}$$

Constant coef. left and exponential right side.  $g(x) = 6 e^{3x}$  a constant times  $e^{-3x}$ 



$$y_{p}' = -3Ae^{-2}$$
  
 $y_{e}'' = 9A e^{3x}$ 

$$y_{p}^{"}-4y_{p}^{+}+4y_{p}=6e^{-3}$$

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$$PAe^{3x} - 4(-3Ae^{3x}) + 4Ae^{3x} = 6e^{-3x}$$

$$QSAe^{3x} = 6e^{-3x}$$
This requires 
$$QSA = 6 \Rightarrow A = \frac{6}{QS}$$
Hence 
$$y_{P} = \frac{6}{QS}e^{-3x}$$

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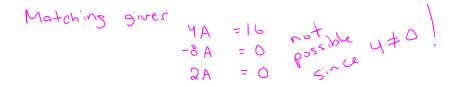
#### Make the form general

$$y^{\prime\prime}-4y^{\prime}+4y=16x^2$$

The left is constant coefficient and the right is a polynomial. Note that  $g(x) = 16x^2$  is a monomial and a polynomial. Suppose we assume  $y_p = Ax^2$  (this work)

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$$y_{P}'' - y_{Y}' + y_{Y} = 16x^{2}$$
  
 $a_{A} - y(z_{Ax}) + y_{Ax^{2}} = 16x^{2}$   
 $z_{P}^{e_{P}}$   
 $y_{Ax^{2}} - y_{Ax} + z_{A} = 16x^{2} + 0x + 0$   
 $z_{P}^{e_{P}}$ 



The correct way to view g(x) = 16x<sup>2</sup> is as a 2<sup>nd</sup> degree polynomial. Then assume yp is also a 2<sup>nd</sup> degree polynomial.

$$y_{p} = A x^{2} + Bx + C$$
Sub 
$$y_{p}' = 2A \times +B$$

$$y_{p}'' = 2A$$

$$y_{p}'' - 4y_{p}' + 4y_{p} = 16 x^{2}$$

$$QA - 4(2Ax + B) + 4(Ax^{2} + Bx + C) = 16x^{2}$$

$$4Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$4Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

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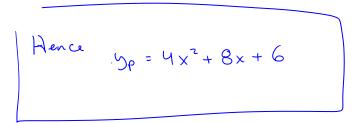
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Solution J A=4  

$$YB = BA \implies B = 2A = B$$
  
 $YC = YB - 2A \implies C = B - \frac{1}{2}A = 8 - \frac{1}{2}(Y) = 6$ 



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### General Form: sines and cosines

$$y''-y'=20\sin(2x)$$

If we assume that  $y_p = A\sin(2x)$ , taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20$$
 and  $-2A = 0$ .

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This is impossible as it would require -5 = 0!

#### General Form: sines and cosines

We must think of our equation  $y'' - y' = 20 \sin(2x)$  as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for  $y_p$  is

$$y_p = A\sin(2x) + B\cos(2x).$$

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## Examples of Forms of $y_p$ based on g (Trial Guesses)

(a) g(x) = 1 (or really any constant) zero degree polynomial  $y_{p} = A$ 

(b) 
$$g(x) = x - 7$$
  
1st degree polynomial  
Sp = A X + B

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## Examples of Forms of $y_p$ based on g (Trial Guesses)

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(c) 
$$g(x) = 5x^2$$
  
 $2^{n2}$  degree polynomial  
 $y_p = Ax^2 + Bx + C$ 

(d) 
$$g(x) = 3x^3 - 5$$
  
 $3^{rd}$  degree polynomial  
 $y_{p} = A x^3 + B x^2 + C x + D$ 

We'll pick this up on Friday.