September 22 Math 2306 sec. 54 Fall 2021

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- ► exponentials, e^{mx} ~
- ► sines and/or cosines, sin(kx), Cos(kx)
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

k- constant

Motivating Example¹

Find a particular solution of the ODE

 We sub this into $y_p'' - 4y_p' + 4y_p = 8x+1$ $y_p = Ax+B$ $y_p' = A$ $y_p'' = 0$ 4Ax + (-4A+4B) = 8x+1

be con match coefficients $\frac{YA \times + (-YA + YB)}{This requires} = \frac{8 \times +1}{YA} = 8$ -YA + YB = 1

Solving A=Z

<ロト < 回 > < 回 > < 三 > < 三 > 三 三

$$YB = 1 + YA \implies B = \frac{1}{4} + A = \frac{1}{4} + 2 = \frac{9}{4}$$

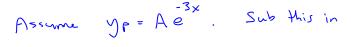
The particular solution is

$$y_p = Zx + \frac{9}{9}$$

< □ ▶ < 圕 ▶ < ≣ ▶ < ≣ ▶ ■ ● ○ Q (℃ September 22, 2021 4/35 The Method: Assume y_p has the same **form** as g(x)

$$y'' - 4y' + 4y = 6e^{-3x}$$

Constant coef. left and exponential right side. $g(x) = 6 e^{3x}$ a constant times e^{-3x}



$$y_{p}' = -3Ae^{-2}$$

 $y_{e}'' = 9A e^{3x}$

$$y_{p}^{"}-4y_{p}^{+}+4y_{p}=6e^{-3}$$

> < 同 > < 回 > < 回 > < 回 > < 回 > < 回 < □ > < □ > < □ < □ </p>

X

$$PAe^{3x} - 4(-3Ae^{3x}) + 4Ae^{3x} = 6e^{-3x}$$

$$QSAe^{3x} = 6e^{-3x}$$
This requires
$$QSA = 6 \Rightarrow A = \frac{6}{QS}$$
Hence
$$y_{P} = \frac{6}{QS}e^{-3x}$$

~

September 22, 2021 6/35

Make the form general

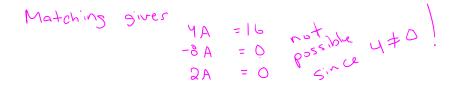
$$y^{\prime\prime}-4y^{\prime}+4y=16x^2$$

The left is constant coefficient and the right is a polynomial. Note that $g(x) = 16x^2$ is a monomial and a polynomial. Suppose we assume $y_p = Ax^2$ (this work)

イロト イポト イヨト 一足

$$y_{P}'' - y_{Y}' + y_{Y} = 16x^{2}$$

 $a_{A} - y(z_{Ax}) + y_{Ax^{2}} = 16x^{2}$
 $z_{P}^{e_{P}}$
 $y_{Ax^{2}} - y_{Ax} + z_{A} = 16x^{2} + 0x + 0$
 $z_{P}^{e_{P}}$



The correct way to view g(x) = 16x² is as a 2nd degree polynomial. Then assume yp is also a 2nd degree polynomial.

$$y_{p} = A x^{2} + Bx + C$$
Sub
$$y_{p}' = 2A \times +B$$

$$y_{p}'' = 2A$$

$$y_{p}'' - 4y_{p}' + 4y_{p} = 16 x^{2}$$

$$QA - 4(2Ax + B) + 4(Ax^{2} + Bx + C) = 16x^{2}$$

$$4Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$4Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (2A - 4B + 4C) = 16x^{2} + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (-8A + 4B) \times + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + (-8A + 4B) \times + 0x + 0$$

$$Ax^{2} + (-8A + 4B) \times + 0x + 0$$

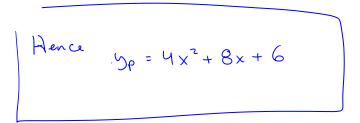
$$Ax^{2} + (-8A + 4B) \times + 0x + 0$$

September 22, 2021 9/35

Solution J A=4

$$YB = BA \implies B = 2A = B$$

 $YC = YB - 2A \implies C = B - \frac{1}{2}A = 8 - \frac{1}{2}(Y) = 6$



(日)

General Form: sines and cosines

$$y''-y'=20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20$$
 and $-2A = 0$.

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > □ ≥
 September 22, 2021

11/35

This is impossible as it would require -5 = 0!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20 \sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

イロト 不得 トイヨト イヨト 二日

September 22, 2021

12/35

Examples of Forms of y_p based on g (Trial Guesses)

(a) g(x) = 1 (or really any constant) zero degree polynomial $y_{p} = A$

(b)
$$g(x) = x - 7$$

1st degree polynomial
Sp = A X + B

< □ ▶ < □ ▶ < ■ ▶ < ■ ▶ < ■ ▶ = つへで September 22, 2021 13/35

Examples of Forms of y_p based on g (Trial Guesses)

イロン イ団 とく ヨン ・ ヨン …

September 22, 2021

3

14/35

(c)
$$g(x) = 5x^2$$

 2^{n2} degree polynomial
 $y_p = Ax^2 + Bx + C$

(d)
$$g(x) = 3x^3 - 5$$

 3^{rd} degree polynomial
 $y_{p} = A x^3 + B x^2 + C x + D$

We'll pick this up on Friday.