September 22 Math 3260 sec. 53 Fall 2025

Chapter 3: Matrix Algebra

For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ and scalar c, we defined

- the row and column vectors,

$$\text{Row}_i(A) = \langle a_{i1}, a_{i2}, \dots, a_{in} \rangle, \quad \text{for} \quad i = 1, \dots, m$$

$$\text{and} \quad \text{Col}_j(A) = \langle a_{1j}, a_{2j}, \dots, a_{mj} \rangle, \quad \text{for} \quad i = j, \dots, n$$

- matrix addition: $A + B = [a_{ij} + b_{ij}]$, and
- scalar multiplication $cA = [ca_{ij}]$.

Recall that the entries can be referenced using two notations,

$$a_{ij} = A_{(i,j)}.$$

3.3 Multiplication of Two Matrices

Suppose A is an $m \times p$ matrix and B is a $p \times n$ matrix. Then the product AB is the $m \times n$ matrix

$$AB = \left[(AB)_{(i,j)}\right], \quad \text{where} \quad (AB)_{(i,j)} = \mathrm{Row}_i(A) \cdot \mathrm{Col}_j(B).$$

If the number of columns of A does not match the number of rows of B, then AB is not defined.

$$A\ B$$

$$m \times \underbrace{p\ p}_{(=)} \times n$$

$$m \times n$$

The inner numbers must match for the product to be defined, and the size of the product is determined by the outer numbers.

Example

$$\text{Let} \quad A = \begin{bmatrix} -4 & -3 & -2 \\ 1 & -4 & -5 \\ 4 & -4 & 3 \\ 6 & 2 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 5 \\ 1 & 5 \end{bmatrix}.$$

Find the product AB.

AB

$4\times 3 \quad 3\times 2$

4×2

$(AB)_{(1,1)} = \text{Row}_1(A)\cdot \text{Col}_1(B)$

$= \langle -4,-3,-2\rangle \cdot \langle 4,0,1\rangle = -18$

$(AB)_{(1,2)} = \text{Row}_1(A)\cdot \text{Col}_2(B)$

$= \langle -4,-3,-2\rangle \cdot \langle -1,5,5\rangle = -21$

$$A = \begin{bmatrix} -4 & -3 & -2 \\ 1 & -4 & -5 \\ 4 & -4 & 3 \\ 6 & 2 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 5 \\ 1 & 5 \end{bmatrix}$$

$$(AB)_{(2,1)} = \langle 1, -4, -5 \rangle \cdot \langle 4, 0, 1 \rangle = -1$$

$$(AB)_{(2,2)} = \langle 1, -4, -5 \rangle \cdot \langle -1, 5, 5 \rangle = -46$$

$$(AB)_{(3,1)} = \langle 4, -4, 3 \rangle \cdot \langle 4, 0, 1 \rangle = 19$$

$$(AB)_{(3,2)} = \langle 4, -4, 3 \rangle \cdot \langle -1, 5, 5 \rangle = -9$$

$$A = \begin{bmatrix} -4 & -3 & -2 \\ 1 & -4 & -5 \\ 4 & -4 & 3 \\ 6 & 2 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 5 \\ 1 & 5 \end{bmatrix}$$

$$(AB)_{(4,1)} = \langle 6, 2, -6 \rangle \cdot \langle 4, 0, 1 \rangle = 18$$

$$(AB)_{(4,2)} = \langle 6, 2, -6 \rangle \cdot \langle -1, 5, 5 \rangle = -26$$

$$= 6(-1) + 2(5) + (-6)(5) = -6 + 10 - 30$$

$$AB = \begin{bmatrix} -18 & -21 \\ -1 & -46 \\ 19 & -9 \\ 18 & -26 \end{bmatrix}$$

Example $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}$

Find the product AB.

$$\underset{2\times 2}{A}\ \underset{2\times 2}{B} \quad \checkmark \quad \downarrow \quad 2\times 2$$

$$AB = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}$

Find the product BA.

$$\underset{2\times 2}{B}\ \underset{2\times 2}{A} \rightarrow 2\times 2$$

$$BA = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -7 \\ 28 & -14 \end{bmatrix}$$

Matrix Multiplication Does Not Commute

- If the product AB is defined, it is not necessarily true that BA is defined.
- If AB and BA are both defined, the products are not necessarily the same size.
- If A and B are both $n \times n$ matrices, then both AB and BA will be defined and will be $n \times n$.
- However, even in this case, in general

$$AB \neq BA.$$

It's not impossible to find a pair of matrices A and B for which $AB = BA$. However, these are special examples.

Example $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}$ and $C = \begin{bmatrix} -5 & 2 \\ -10 & 4 \end{bmatrix}$

We computed $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Now, compute AC.

$A \quad C$

$2\times 2 \quad 2\times 2$ ✓

$\downarrow$

2×2

$$AC = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ -10 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

More Caveats

The zero product property of real numbers.

If a and b are real numbers such that $ab = 0$, then $a = 0$ or $b = 0$.

Question: If A and B are 2×2 matrices such that $AB = O_{2\times 2}$, can we conclude that $A = O_{2\times 2}$ or $B = O_{2\times 2}$?

No, AB can equal $O_{2\times 2}$ even

if $A \neq O_{2\times 2}$ and $B \neq O_{2\times 2}$.

More Caveats

Cancelation law of real numbers.

If a, b and c are nonzero real numbers such that $ab = ac$, then $b = c$. That is, a cancels.

Question: If A, B, and C are 2×2 matrices such that $AB = AC$, can we conclude that $B = C$? That is, does A cancel?

No, AB can equal AC even if $B \neq C$.

Consequences of Non-commutativity

- There is no "*zero product property*" for matrix multiplication. That is, $AB = O_{m\times n}$ **DOES NOT** imply that A or B is a zero matrix.
- There is no "*cancelation law*" for matrix multiplication. That is, $AB = AC$ **DOES NOT** imply that $B = C$.

Exercise: Let $X = \begin{bmatrix} 4 & 1 \\ -12 & -3 \end{bmatrix}$. Compute $X^2 = XX$.

$$X^2 = XX = \begin{bmatrix} 4 & 1 \\ 12 & -3 \end{bmatrix}$$

X is called Idempotent.

Algebraic Properties

Some Algebraic Properties

Suppose A is an $m \times p$ matrix and B and C are $p \times n$ matrices. Then
$$A(B + C) = AB + AC.$$

And, if c is any scalar, then

$$A(cB) = (cA)B = c(AB).$$

Remark: It is also true that matrix multiplication distributes on the right side of matrix addition. That is, if A and B are $m \times p$ and C is $p \times n$, then
$$(A + B)C = AC + BC.$$

3.4 The Transpose of a Matrix

Transpose

Suppose $A = [a_{ij}]$ is an $m \times n$ matrix. The matrix A^T, called the **transpose** of A, is the $n \times m$ matrix defined by

$$\left(A^T\right)_{(i,j)} = A_{(j,i)}.$$

That is,

$$A = [a_{ij}] \quad \Longleftrightarrow \quad A^T = [a_{ji}].$$

Note that this implies that

$$\mathrm{Row}_i(A) = \mathrm{Col}_i(A^T) \quad \text{and} \quad \mathrm{Col}_j(A) = \mathrm{Row}_j(A^T).$$

Example

Identify A^T and B^T given

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

$\underset{2\times 2}{A}$ $\underset{2\times 2}{A^T}$ $A^T = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

$\underset{2\times 3}{B}$ $\underset{3\times 2}{B^T}$ $B^T = \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2 & 6 \end{bmatrix}$

Examples

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate $\left(A^T\right)^T$

$$A^T = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$$

$$(A^T)^T = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} = A$$

Examples

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate AB and $(AB)^T$ or state why they are not defined.

AB

$2\times 2 \quad 2\times 3$

$\Downarrow$

2×3

$$AB = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} -1 & -2 \\ 12 & -8 \\ -16 & 8 \end{bmatrix}$$

Examples

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate BA and $(BA)^T$ or state why they are not defined.

$B \quad A$

$2\times 3 \quad 2\times 2$

$\neq$

BA is not defined,

B has 3 columns

A has 2 rows.

Examples

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate $B^T A^T$ or state why it is not defined.

$$\underset{3\times 2}{B^T} \ \underset{2\times 2}{A^T}$$

$$\downarrow$$

$$3\times 2$$

$$B^T A^T = \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 12 & -8 \\ -16 & 8 \end{bmatrix}$$

Examples

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate $A^T B^T$ or state why it is not defined.

$A^T \quad B^T$

$2\times 2 \quad 3\times 2$

$\neq$

$A^T B^T$ is not defined

A^T has 2 columns,

B^T has 3 rows.

Observation

For this example, we found that $(AB)^T = B^T A^T$. This is not a coincidence. We can argue that

$$\left((AB)^T\right)_{(i,j)} = \left(B^T A^T\right)_{(i,j)}.$$

Recall: $X^T_{(p,q)} = X_{(q,p)}$

$(XY)_{(p,q)} = \text{Row}_p(X) \cdot \text{Col}_q(Y)$

$\text{Row}_p(X) = \text{Col}_p(X^T)$ and

$\vec{x} \cdot \vec{y} = \vec{y} \cdot x$

These are the four properties needed for our proof.

$$\left((AB)^T\right)_{(i,j)} = \left(B^T A^T\right)_{(i,j)}$$

$$\left((AB)^T\right)_{(i,j)} = (AB)_{(j,i)}$$

$$= \text{Row}_j(A) \cdot \text{Col}_i(B)$$

$$= \text{Col}_j(A^T) \cdot \text{Row}_i(B^T)$$

$$= \text{Row}_i(B^T) \cdot \text{Col}_j(A^T)$$

$$= (B^T A^T)_{(i,j)}$$

Since each entry of $(AB)^T$ is equal to the corresponding entry of $B^T A^T$, we can conclude that $(AB)^T = B^T A^T$.

Algebraic Properties

Let A and B be matrices such that the appropriate sums and products are defined, and let r be a scalar. Then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$

Remark: Note what this last property says. It says that the transpose of a product is the product of the transposes, but in the reverse order. This can be extended to a product of more than two matrices. For example, when the products are defined,

$$(ABC)^T = C^T B^T A^T, \quad \text{and} \quad (ABCDE)^T = E^T D^T C^T B^T A^T, \quad \text{etc.}$$