September 23 Math 2306 sec. 51 Fall 2022

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order¹, linear, homogeneous equation with constant coefficients

$$arac{d^2y}{dx^2}+brac{dy}{dx}+cy=0, \quad ext{with } a
eq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\frac{dy}{dx} - \frac{c}{a}y.$$

Question: What sorts of functions y could be expected to satisfy

$$y'' = (\text{constant}) y' + (\text{constant}) y?$$

 $y = e^{mx} (-\cos)^{mx} (\sin e^{-5}) (\cos^{-5} e^{-5})$

¹We'll extend the result to higher order at the end of this sectionSeptember 21, 2022 1/24

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We look for solutions of the form $y = e^{mx}$ with m constant.

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 $am^2 + bm + C = 0$

y= e will solve the ODE if the number in solver this graduatic equation.

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Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

 $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

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Case I: Two distinct real roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac > 0$.

There are two different roots m_1 and m_2 . A fundamental solution set consists of

$$y_1 = e^{m_1 x}$$
 and $y_2 = e^{m_2 x}$.

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

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Example

Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$

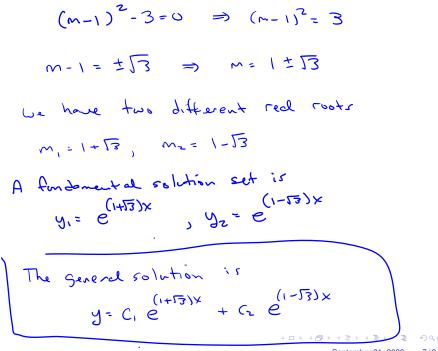
The opt is homogeneous a) constant
coefficients.
The Characteristic equation is
$$m^{2} - 2m - 2 = 0$$

Find the roots. Completing the square
$$m^{2} - 2m + 1 - 1 - 2 = 0$$

perfect square
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Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac = 0$

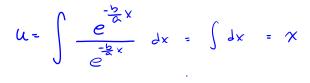
There is only one real, double root, $m = \frac{-b}{2a}$.

Use reduction of order to find the second solution to the equation (in standard form)

 $y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$ $y_2 = (4y_1), \quad \text{where} \quad u = \int \frac{-\int r \cos 3x}{(y_1)^2} \, dx$ Here $P(x) = \frac{b}{a}, \quad -\int r \cos 3x = e^{-\frac{b}{a}x} = e^{-\frac{b}{a}x}$

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$$(y_1)^2 = \left(\begin{array}{c} e^{-\frac{b}{2a}\chi} \\ e^{-\frac{b}{2a}\chi} \end{array}\right)^2 = e^{-\frac{2b}{2a}\chi} = e^{-\frac{b}{2a}\chi}$$



 $\Rightarrow y_2 = x e^{-\frac{b}{z_A}x}$

Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac = 0$

If the characteristic equation has one real repeated root *m*, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx}$$
 and $y_2 = xe^{mx}$.

The general solution is

$$y=c_1e^{mx}+c_2xe^{mx}.$$

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Case III: Complex conjugate roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac < 0$

The two roots of the characteristic equation will be

$$m_1 = \alpha + i\beta$$
 and $m_2 = \alpha - i\beta$ where $i^2 = -1$.

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We want our solutions in the form of <u>real valued</u> functions. We start by writing a pair of solutions

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}$$
, and $Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}$.

We will use the **principle of superposition** to write solutions y_1 and y_2 that do not contain the complex number *i*.

Deriving the solutions Case III

Recall Euler's Formula² : $e^{i\theta} = \cos \theta + i \sin \theta$.

$$Y_{1} = e^{\alpha x} e^{i\beta x} = e^{\alpha x} \left(\cos \left(\beta x \right) + i \sin \left(\beta x \right) \right)$$

$$Y_{2} = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}\left(\cos\left(\beta x\right) - i\sin\left(\beta x\right)\right)$$

$$Y_{1} = e^{\alpha x}\cos\left(\beta x\right) + ie^{\alpha x}\sin\left(\beta x\right)$$

$$Y_{2} = e^{\alpha x}\cos\left(\beta x\right) - ie^{\alpha x}\sin\left(\beta x\right)$$

$$Y_{2} = e^{\alpha x}\cos\left(\beta x\right) - ie^{\alpha x}\sin\left(\beta x\right)$$

$$S_{2}x = y_{1} = \frac{1}{2}(y_{1} + y_{2}) \quad a \neq y_{2} = \frac{1}{2i}\left(y_{1} - y_{2}\right)$$
²As the sine is an odd function $e^{-i\theta} = \cos\theta - i\sin\theta$.

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 $y_1 = \frac{1}{2} \left(Z e^{\alpha \times} Cos(\beta \times) \right) = e^{\alpha \times} Cos(\beta \times)$ $y_2 = \frac{1}{z_i} \left(2i e^{ix} Sim(\beta x) \right) = e^{ix} Sim(\beta x)$ The fundamental solution set is $y_1 = e^{\alpha x} G_2(\beta x)$, $y_2 = e^{\alpha x} S_2(\beta x)$

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Case III: Complex conjugate roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac < 0$

Let α be the real part of the complex roots and β be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x} \cos(\beta x)$$
 and $y_2 = e^{\alpha x} \sin(\beta x)$.

The general solution is

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x).$$

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