September 23 Math 2306 sec. 51 Fall 2022

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \quad \text{with } a \neq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\frac{dy}{dx} - \frac{c}{a}y.$$

Question: What sorts of functions y could be expected to satisfy

$$y'' = (\text{constant})\, y' + (\text{constant})\, y?$$

$y = e^{mx}$ m - constant sines | cosines

[1] We'll extend the result to higher order at the end of this section.

We look for solutions of the form $y = e^{mx}$ with m constant.

$$ay'' + by' + cy = 0$$

If $y = e^{mx}$, $y' = me^{mx}$ and $y'' = m^2 e^{mx}$.

Substitute

$$a(m^2 e^{mx}) + b(me^{mx}) + ce^{mx} = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

This holds for all x in some interval.

This will be true if m satisfies

$$am^2 + bm + c = 0$$

$y = e^{mx}$ will solve the ODE if the number m solves this quadratic equation.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Example

Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$

The ODE is homogeneous w/ constant coefficients.

The characteristic equation is

$$m^2 - 2m - 2 = 0$$

Find the roots. Completing the square

$$\underbrace{m^2 - 2m + 1}_{\text{perfect square}} - 1 - 2 = 0$$

$$(m-1)^2 - 3 = 0 \Rightarrow (m-1)^2 = 3$$

$$m - 1 = \pm\sqrt{3} \Rightarrow m = 1 \pm \sqrt{3}$$

We have two different real roots

$$m_1 = 1 + \sqrt{3}, \quad m_2 = 1 - \sqrt{3}$$

A fundamental solution set is

$$y_1 = e^{(1+\sqrt{3})x}, \quad y_2 = e^{(1-\sqrt{3})x}$$

The general solution is

$$y = C_1 e^{(1+\sqrt{3})x} + C_2 e^{(1-\sqrt{3})x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

There is only one real, double root, $m = \dfrac{-b}{2a}$.

Use reduction of order to find the second solution to the equation (in standard form)

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$$

$y_2 = uy_1$ where $u = \displaystyle\int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$

Here $P(x) = \frac{b}{a}$, $e^{-\int P(x)\,dx} = e^{-\int \frac{b}{a}\,dx} = e^{-\frac{b}{a}x}$

$$(y_1)^2 = \left(e^{-\frac{b}{2a}x}\right)^2 = e^{-\frac{2b}{2a}x} = e^{-\frac{b}{a}x}$$

$$u = \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx = \int dx = x$$

$$\Longrightarrow \quad y_2 = x\, e^{-\frac{b}{2a}x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 xe^{mx}.$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

The two roots of the characteristic equation will be

$$m_1 = \alpha + i\beta \quad \text{and} \quad m_2 = \alpha - i\beta \quad \text{where} \quad i^2 = -1.$$

$\beta > 0$

We want our solutions in the form of *real valued* functions. We start by writing a pair of solutions

$$Y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x}.$$

We will use the **principle of superposition** to write solutions y_1 and y_2 that do not contain the complex number i.

Deriving the solutions Case III

Recall Euler's Formula[2] : $e^{i\theta} = \cos\theta + i\sin\theta$.

$$Y_1 = e^{\alpha x}e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$Y_2 = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

$$Y_1 = e^{\alpha x}\cos(\beta x) + i\,e^{\alpha x}\sin(\beta x)$$

$$Y_2 = e^{\alpha x}\cos(\beta x) - i\,e^{\alpha x}\sin(\beta x)$$

Set $y_1 = \frac{1}{2}(Y_1 + Y_2)$ and $y_2 = \frac{1}{2i}(Y_1 - Y_2)$

[2] As the sine is an odd function $e^{-i\theta} = \cos\theta - i\sin\theta$.

$$y_1 = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$$

$$y_2 = \frac{1}{2i}\left(2i\, e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$$

The fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x)$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

Let α be the real part of the complex roots and β be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$