September 23 Math 2306 sec. 51 Fall 2024

Section 7: Reduction of Order

- We start with a second order, linear, homogeneous ODE in standard form
$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$
- We know one solution y_1. (Keep in mind that y_1 is a known!)
- We try to find a second **linearly independent solution** y_2 by guessing that it can be found in the form
$$y_2(x) = u(x)y_1(x)$$
where the goal becomes finding u.
- **Due to linear independence, we know that u cannot be constant.**

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - \text{is known.}$$

Let $y_2 = u(x)y_1(x)$. y_2 is supposed to solve the ODE; so substitute it in.

$$y_2 = y_1 u$$

$$y_2' = y_1 u' + y_1' u$$

$$y_2'' = y_1 u'' + y_1' u' + y_1' u' + y_1'' u$$

We know that $\quad y_1'' + P(x)y_1' + Q(x)y_1 = 0.$

$$y_2'' = y_1 u'' + 2y_1' u' + y_1'' u$$

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0,$$

$$y_2'' + P(\;.\; y_2' + Q(x)y_2 = 0$$

$$y_1u'' + 2y_1'u' + y_1''u + P(x)\left(y_1u' + y_1'u\right) + Q(x)\left(y_1u\right) = 0$$

Collect u'', u', u

$$y_1u'' + \left(2y_1' + P(x)y_1\right)u' + \underbrace{\left(y_1'' + P(x)y_1' + Q(x)y\right)}_{=0}u = 0$$

The equation reduces to

$$y_1u'' + \left(2y_1' + P(x)y_1\right)u' = 0$$

Let $w = u'$, then $w' = u''$, and w solves

$$y_1 w' + (2y_1' + P(x) y_1) w = 0$$

This is 1^{st} order linear and separable.

Separate variables.

$$y_1 \frac{dw}{dx} = -\left(2\frac{dy_1}{dx} + P(x) y_1\right) w$$

$$\frac{1}{w}\frac{dw}{dx} = -2\frac{\frac{dy_1}{dx}}{y_1} - P(x)$$

$$\frac{1}{w}\, dw = -2\frac{dy_1}{y_1} - P(x)\, dx$$

$$\int \frac{1}{w}\,dw = -2\int \frac{dy_1}{y_1} - \int P(x)\,dx$$

$$\ln w = -2\ln|y_1| - \int P(x)\,dx$$

$$e^{\ln w} = e^{\ln y_1^{-2} - \int P(x)\,dx}$$

$$w = y_1^{-2}\, e^{-\int P(x)\,dx}$$

$$w = \frac{e^{-\int P(x)\,dx}}{y_1^2}.$$

Since $w = u'$, we get

$$u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2(x) = y_1(x)u(x) \quad \text{where} \quad u(x) = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the general solution of the differential equation for which one solution is known.

$$x^2y'' + xy' + y = 0, \quad x > 0, \quad y_1(x) = \cos(\ln x)$$

$y_2 = uy_1$ where $u = \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$

Put the ODE in standard form:

$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0 \qquad P(x) = \frac{1}{x}$

$-\int p(x)\,dx = -\int \frac{1}{x}\,dx = -\ln x = \ln x^{-1}$

$$u = \int \frac{e^{\ln x^{-1}}}{(\cos(\ln x))^2}\,dx = \int \frac{x^{-1}}{\cos^2(\ln x)}\,dx$$

$$= \int \sec^2(\ln x)\,\frac{1}{x}\,dx$$

let $v = \ln x$

$dv = \frac{1}{x}\cdot dx$

$$= \int \sec^2 v\,dv = \tan v$$

$$u = \tan(\ln x).$$

$$y_2 = u y_1 = \tan(\ln x)\cos(\ln x) = \sin(\ln x)$$

$$y_1 = \cos(\ln x), \quad y_2 = \sin(\ln x)$$

The general solution is

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 1$$

Look for $y_2 = u y_1$, where $u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$

$P(x) = 4$, $\quad -\int P(x)\,dx = -\int 4\,dx = -4x$.

$$u = \int \frac{e^{-4x}}{(e^{-2x})^2}\,dx = \int \frac{e^{-4x}}{e^{-4x}}\,dx$$

$$= \int dx = x$$

$y_2 = u y_1 = x e^{-2x}$

The general solution $y = c_1 e^{-2x} + c_2 x e^{-2x}$.

Apply $y(0) = 1$ and $y'(0) = 1$.

$y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$

$y(0) = c_1 e^0 + c_2 (0) e^0 = 1 \Rightarrow c_1 = 1$

$y'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 (0) e^0 = 1$

$-2c_1 + c_2 = 1$

$\Rightarrow c_2 = 1 + 2c_1 = 1 + 2(1) = 3$

The solution to the IVP is

$$y = e^{-2x} + 3x e^{-2x}$$