September 23 Math 2306 sec. 52 Fall 2022

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0, \quad \text{with } a\neq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2}=-\frac{b}{a}\frac{dy}{dx}-\frac{c}{a}y.$$

Question: What sorts of functions y could be expected to satisfy

$$y''=(\text{constant})\,y'+(\text{constant})\,y?$$

$y=e^{mx}$ m - constant

sines / cosines

[1] We'll extend the result to higher order at the end of this section.

We look for solutions of the form $y = e^{mx}$ with m constant.

$$ay'' + by' + cy = 0$$

Let $y = e^{mx}$, then $y' = me^{mx}$, $y'' = m^2 e^{mx}$

Substitute into the ODE

$$a(m^2 e^{mx}) + b(me^{mx}) + c(e^{mx}) = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

This is to hold for all x in some interval.

This will be true if m solves the

polynomial equation

$$am^2 + bm + C = 0$$

If m solves this, then $y = e^{mx}$

solves the ODE.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Example

Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$

This is linear, homogeneous, w/ constant coefficients.

The characteristic equation is

$$m^2 - 2m - 2 = 0$$

Find the roots:

complete the square $\underbrace{m^2 - 2m + 1}_{\text{perfect square}} - 1 - 2 = 0$

$(m-1)^2 - 3 = 0 \implies (m-1)^2 = 3$

$m - 1 = \pm\sqrt{3} \implies m = 1 \pm \sqrt{3}$

2 different roots $m_1 = 1+\sqrt{3}$, $m_2 = 1-\sqrt{3}$

The fundamental solution set is

$y_1 = e^{(1+\sqrt{3})x}$, $y_2 = e^{(1-\sqrt{3})x}$

The general solution is

$$\boxed{y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

There is only one real, double root, $m = \dfrac{-b}{2a}$.

Use reduction of order to find the second solution to the equation (in standard form)

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$$

$y_2 = u y_1$ where $u = \displaystyle\int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$

$P(x) = \dfrac{b}{a}$, $-\int P(x)\,dx = -\int \dfrac{b}{a}\,dx = -\dfrac{b}{a}x$

$$e^{-\int P(x)\,dx} = e^{-\frac{b}{a}x}, \quad (y_1)^2 = \left(e^{\frac{-b}{2a}x}\right)^2 = e^{-\frac{2b}{2a}x}$$

$$= e^{-b/a\, x}$$

$$u = \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx = \int dx = x$$

$$y_2 = u y_1 = x\, e^{\frac{-b}{2a}x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 xe^{mx}.$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

The two roots of the characteristic equation will be

$$m_1 = \alpha + i\beta \quad \text{and} \quad m_2 = \alpha - i\beta \quad \text{where} \quad i^2 = -1.$$

$\beta > 0$

We want our solutions in the form of *real valued* functions. We start by writing a pair of solutions

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

We will use the **principle of superposition** to write solutions y_1 and y_2 that do not contain the complex number i.

Deriving the solutions Case III

Recall Euler's Formula[2] : $e^{i\theta} = \cos\theta + i\sin\theta$.

$$Y_1 = e^{\alpha x}e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$Y_2 = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

$$Y_1 = e^{\alpha x}\cos(\beta x) + i\,e^{\alpha x}\sin(\beta x)$$

$$Y_2 = e^{\alpha x}\cos(\beta x) - i\,e^{\alpha x}\sin(\beta x)$$

$$y_1 = \frac{1}{2}(Y_1 + Y_2) \quad \text{and} \quad y_2 = \frac{1}{2i}(Y_1 - Y_2)$$

[2] As the sine is an odd function $e^{-i\theta} = \cos\theta - i\sin\theta$.

$$y_1 = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$$

$$y_2 = \frac{1}{2i}\left(2i\, e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$$

$\{y_1, y_2\}$ will be our fundamental solution set.

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

Let α be the real part of the complex roots and β be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Example

Find the general solution of $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 6x = 0.$

Linear, homogeneous, constant coef.

Char. eqn. $m^2 + 4m + 6 = 0$

Find m: $m = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(6)}}{2}$

$= \dfrac{-4 \pm \sqrt{-8}}{2} = \dfrac{-4 \pm 2\sqrt{2}\,i}{2}$

$m = -2 \pm \sqrt{2}\, i$

Complex w/ $\alpha = -2$ and $\beta = \sqrt{2}$

$$x_1 = e^{-2t}\cos(\sqrt{2}\,t) \quad , \quad x_2 = e^{-2t}\sin(\sqrt{2}\,t)$$

The general solution is

$$x = C_1 e^{-2t}\cos(\sqrt{2}\,t) + C_2 e^{-2t}\sin(\sqrt{2}\,t)$$