September 23 Math 2306 sec. 53 Fall 2024

Section 7: Reduction of Order

We start with a second order, linear, homogeneous ODE in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

- ▶ We know one solution y_1 . (Keep in mind that y_1 is a known!)
- ▶ We try to find a second **linearly independent solution** y_2 by guessing that it can be found in the form

$$y_2(x) = u(x)y_1(x)$$

where the goal becomes finding u.

Due to linear independence, we know that u cannot be constant.

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - \text{is known.}$$
Let $y_2 = u(x)y_1(x)$. This is supposed to solve

the ODF. Suib by into the ODE.

$$y_z = y_1 u$$

 $y_z' = y_1 u' + y_1' u$
 $y_z'' = y_1 u'' + y_1' u' + y_1' u' + y_1'' u$
We know that $y_1'' + P(x)y_1' + Q(x)y_1 = 0$.
 $y_z'' = y_1 u'' + Zy_1' u' + y_1'' u$

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0,$$

$$y_{z}^{(1)} + P(x)y_{z}^{(1)} + Q(x)y_{z}^{(2)} = 0$$

$$\underline{y},\underline{u}'+z\underline{y},\underline{u}'+\underline{y},\underline{u}+Px(\underline{y},\underline{u}'+\underline{y},\underline{u})+Qx(\underline{y},\underline{u})=0$$

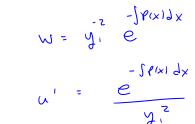
This is 1st order Inea and separable

Separate variouses
$$y_1 \frac{dw}{dx} = -\left(2 \frac{dy_1}{dx} + P(x)y_1\right) U$$

$$\frac{1}{w} \frac{dw}{dx} = -2 \frac{dy_1}{dx} - P(x)$$

$$\frac{1}{w} cw = -2 \frac{dy_1}{dx} dx - P(x) dx$$

$$\int \frac{1}{w} dw = -2 \frac{dy_1}{dx} - P(x) dx$$



$$u = \int \frac{-\int e(x)dx}{e} dx$$

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Reduction of Order Formula

For the second order, homogeneous equation in standard form with one known solution y_1 , a second linearly independent solution y_2 is given by

$$y_2(x) = y_1(x)u(x)$$
 where $u(x) = \int \frac{e^{-\int P(x) dx}}{(y_1(x))^2} dx$

Example

Find the general solution of the differential equation for which one solution is known.

$$x^{2}y'' + xy' + y = 0, \quad x > 0, \quad y_{1}(x) = \cos(\ln x)$$

$$we'll \quad find \quad y_{2} \quad as \quad y_{2} = uy, \quad where$$

$$u = \int \frac{e^{-\int P(x) dx}}{y_{1}^{2}} dx \quad P(x) = \chi \quad ? \quad No'.$$

$$\ln stadard \quad form \quad the \quad OPE \quad is$$

$$y'' + \frac{1}{\lambda}y' + \frac{1}{\lambda^{2}}y = 0 \quad P(x) = \frac{1}{\lambda}$$

$$-\int P(x) dx = -\int \frac{1}{\lambda} dx = -\ln x, \quad e^{-\int P(x) dx} = \frac{1}{\lambda^{2}}$$

$$u = \int \frac{x^{-1}}{(C_{0}s \ln x)^{2}} dx = \int Scc^{2}(\ln x) \frac{1}{x} dx$$

$$let \quad v = \ln x, \quad dv = \frac{1}{x} dx$$

$$= \int Sec^{2}v dv = tonv = ton(\ln x)$$

$$y_{1} = C_{1}(\ln x), \quad v = ton(\ln x), \quad y_{2} = uy,$$

$$y_{2} = ton(\ln x) C_{1}(\ln x) = S_{1}(\ln x)$$

$$y_{3} = ton(\ln x) C_{2}(\ln x) = S_{1}(\ln x)$$

$$y_{4} = C_{1}(\ln x) C_{2}(\ln x) + C_{2}(\ln x)$$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 1$$
First find $y_2 = uy$, when $u = \begin{cases} \frac{-\int P(x) dx}{y_1^2} dx \end{cases}$

$$P(x) = 4 \quad 7 \quad y = 0$$

$$-\int P(x) dx = -\int 4 dx = -4x$$

$$u = \begin{cases} \frac{-4x}{e^{-4x}} dx = \int \frac{-4x}{e^{-4x}} dx = \int dx = x \end{cases}$$

$$y_1 = e^{2x}$$
, $y_2 = x e^{x}$ is a fundamental solution is

 $y = c_1 e^{2x} + c_2 x e^{-2x}$

Apply $y(0) = 1$ and $y'(0) = 1$
 $y' = -2c_1 e^{2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$
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The solution to the IVP is
$$y = e^{-2x} + 3x e^{-2x}$$