September 23 Math 2306 sec. 53 Fall 2024

Section 7: Reduction of Order

- We start with a second order, linear, homogeneous ODE in standard form
$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$
- We know one solution y_1. (Keep in mind that y_1 is a known!)
- We try to find a second **linearly independent solution** y_2 by guessing that it can be found in the form
$$y_2(x) = u(x)y_1(x)$$
where the goal becomes finding u.
- **Due to linear independence, we know that u cannot be constant.**

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - \text{is known.}$$

Let $y_2 = u(x)y_1(x)$. This is supposed to solve the ODE. Sub y_2 into the ODE.

$y_2 = y_1 u$

$y_2' = y_1 u' + y_1' u$

$y_2'' = y_1 u'' + y_1' u' + y_1' u' + y_1'' u$

We know that $\quad y_1'' + P(x)y_1' + Q(x)y_1 = 0.$

$y_2'' = y_1 u'' + 2y_1' u' + y_1'' u$

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0,$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

$$y_1u'' + 2y_1'u' + y_1''u + P(x)(y_1u' + y_1'u) + Q(x)(y_1u) = 0$$

Collect u, u' and u

$$y_1u'' + (2y_1' + P(x)y_1)u' + (\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{=0})u = 0$$

The equation reduces to

$$y_1u'' + (2y_1' + P(x)y_1)u' = 0$$

Let $w = u'$, then $w' = u''$ and w solves

$$y_1 w' + (2y_1' + P(x) y_1) w = 0$$

This is 1st order linear and separable.

Separate variables

$$y_1 \frac{dw}{dx} = -\left(2 \frac{dy_1}{dx} + P(x) y_1\right) w$$

$$\frac{1}{w} \frac{dw}{dx} = -2 \frac{\frac{dy_1}{dx}}{y_1} - P(x)$$

$$\frac{1}{w} dw = -2 \frac{\frac{dy_1}{dx}}{y_1} dx - P(x)\, dx$$

$$\int \frac{1}{w} dw = \int -2 \frac{dy_1}{y} - \int P(x)\, dx$$

$$\ln w = -2\ln|y_1| - \int p(x)\,dx$$

$$\ln w = \ln y_1^{-2} - \int p(x)\,dx$$

$$e^{\ln w} = e^{\ln y_1^{-2} - \int p(x)\,dx} = e^{\ln y_1^{-2}} \cdot e^{-\int p(x)\,dx}$$

$$w = y_1^{-2}\, e^{-\int p(x)\,dx}$$

$$u' = \frac{e^{-\int p(x)\,dx}}{y_1^{2}}$$

$$u = \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2(x) = y_1(x)u(x) \quad \text{where} \quad u(x) = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the general solution of the differential equation for which one solution is known.

$$x^2y'' + xy' + y = 0, \quad x > 0, \quad y_1(x) = \cos(\ln x)$$

we'll find y_2 as $y_2 = uy_1$ where

$$u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx \qquad P(x) = x \;? \quad \text{NO!}$$

In standard form the ODE is

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0 \qquad P(x) = \frac{1}{x}$$

$$-\int P(x)\,dx = -\int \frac{1}{x}\,dx = -\ln x, \quad e^{-\int P(x)dx} = e^{\ln x^{-1}} = x^{-1}$$

$$u = \int \frac{x^{-1}}{(\cos \ln x)^2}\, dx = \int \sec^2(\ln x)\, \frac{1}{x}\, dx$$

Let $v = \ln x$, $dv = \frac{1}{x}\, dx$

$$= \int \sec^2 v\, dv = \tan v = \tan(\ln x)$$

$y_1 = \cos(\ln x)$, $u = \tan(\ln x)$, $y_2 = u y_1$

$$y_2 = \tan(\ln x)\cos(\ln x) = \sin(\ln x)$$

The general solution is

$$\boxed{y = c_1 \cos(\ln x) + c_2 \sin(\ln x)}$$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 1$$

First, find $y_2 = u y_1$ where $u = \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx$

$P(x) = 4$? yes

$-\int P(x)\,dx = -\int 4\,dx = -4x$

$u = \int \frac{e^{-4x}}{(e^{-2x})^2}\,dx = \int \frac{e^{-4x}}{e^{-4x}}\,dx = \int dx = x$

$y_1 = e^{-2x}$, $y_2 = x e^{-2x}$ is a fundamental solution set. The general solution is

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

Apply $y(0) = 1$ and $y'(0) = 1$

$$y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$$

$$y(0) = c_1 e^0 + c_2 (0) e^0 = 1 \implies c_1 = 1$$

$$y'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 (0) e^0 = 1$$

$$-2c_1 + c_2 = 1$$

$C_2 = 1 + 2C_1 = 1 + 2(1) = 3$

The solution to the IVP is

$$\boxed{y = e^{-2x} + 3x e^{-2x}}$$