September 24 Math 2306 sec. 51 Fall 2021

Section 9: Method of Undetermined Coefficients

We were considering linear, constant coefficient, nonhomogeneous ODEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e.g. e^{kx}
- sines and/or cosines, e.g. $\sin(kx)$ or $\cos(kx)$ k-constant
- and products and sums of the above kinds of functions

At first, we are looking at the y_p part. The general solution will be $y = y_c + y_p$.

Method of Undetermined Coefficients

This is a method for finding a particular solution to

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

by assuming that y_p is the same kind of function as g. We

- Determine what type of function g is,
- set up a guess for y_p of this form with unspecified constant coefficients,
- substitute our guess into the ODE,
- and then solve a system of equations for the coefficients by matching like terms.

Examples of Forms of y_p based on g (Trial Guesses)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

(e) $g(x) = xe^{3x}$ 1st degree polynomial times e^{3x}

$$y_p = (Ax + B)e^{3x} = Axe^{3x} + Be^{3x}$$

(f) $g(x) = \cos(7x)$ Linear combo of $\sin(7x)$ & $\cos(7x)$

$$y_p = A\cos(7x) + B\sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$

Linear combos of $\sin(2x)$ and $\cos(2x)$ and
$\sin(4x)$ and $\cos(4x)$

$$y_p = A\cos(2x) + B\sin(2x) + C\cos(4x) + D\sin(4x)$$

(h) $g(x) = x^2\sin(3x)$

2nd-degree poly times $\sin(3x)$ and $\cos(3x)$

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$ $\quad$ e^x times linear combo of $\cos(2x)$ and $\sin(2x)$

$$y_p = A e^x \cos(2x) + B e^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$ $\quad$ 1st degree poly times e^{-x} times linear combo of $\sin(\pi x)$ and $\cos(\pi x)$

$$y_p = (Ax + B) e^{-x} \sin(\pi x) + (Cx + D) e^{-x} \cos(\pi x)$$

The Superposition Principle

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

The principle of superposition for nonhomogeneous equations tells us that we can find y_p by considering separate problems

$$y_{p_1} \text{ solves} \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x)$$

$$y_{p_2} \text{ solves} \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_2(x),$$

and so forth.

Then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$.

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

We can break this into two problems

Find y_{p_1} that solves

$$y'' - 4y' + 4y = 6e^{-3x} \qquad g_1(x) = 6e^{-3x}$$

and Find y_{p_2} that solves

$$y'' - 4y' + 4y = 16x^2 \qquad g_2(x) = 16x^2$$

For $g_1(x) = 6e^{-3x}$, $y_{p_1} = Ae^{-3x}$

For $g_2(x) = 16x^2$, $y_{p_2} = Bx^2 + Cx + D$

Then $y_p = Ae^{-3x} + Bx^2 + Cx + D$

Using the results from Wednesday (9/22/21), we'd find

$$y_p = \frac{6}{25}e^{-3x} + 4x^2 + 8x + 6$$

A Glitch!

$$y'' - y' = 3e^x$$

Constant coef left, Exponential right.

$g(x) = 3e^x$ constant times e^x

Set $y_p = Ae^x$

Trying to find A:

$y_p = Ae^x$

$y_p' = Ae^x$

$y_p'' = Ae^x$

we need $$y_p'' - y_p' = 3e^x$$

$$Ae^x - Ae^x = 3e^x$$

$$0 = 3e^x$$

There is no choice of A for which this is true.

The problem is that our guess for y_p matches at least some term from y_c.

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case I: The guess for y_{p_i} **DOES NOT** have any like terms in common with y_c.

Then our guess for y_{p_i} will work as written. We do the substitution to find the A, B, etc.

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case II: The guess for y_{p_i} **DOES** have a like term in common with y_c.

Then we multiply our guess at y_{p_i} by x^n where n is the smallest positive integer such that our new guess $x^n y_{p_i}$ does not have any like terms in common with y_c. Then we take this new guess and substitute to find the A, B, etc.

Case II Examples

Find the general solution of the ODE.

$y = y_c + y_p$

$$y'' - 2y' + y = -4e^x$$

The left is constant coef. and the right is an exponential.

Find y_c: solve $y'' - 2y' + y = 0$

The characteristic eqn is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0 \implies m = 1 \text{ double root}$$

$$y_1 = e^x, \quad y_2 = xe^x \qquad y_c = c_1 e^x + c_2 x e^x$$

Now find y_p: $g(x) = -4e^x$ constant times e^x.

Start w/ $y_p = Ae^x$ × matches y_1

modify $y_p = (Ae^x)x = Axe^x$ × matches y_2

again $y_p = (Ae^x)x^2 = Ax^2e^x$ ✓ Doesn't match y_c

Sub into $y_p'' - 2y_p' + y_p = -4e^x$

$$y_p = Ax^2e^x$$

$$y_p' = Ax^2e^x + 2Axe^x$$

$$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$$

$$= Ax^2e^x + 4Axe^x + 2Ae^x$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = -4e^x$$

Collect x^2e^x, xe^x, e^x

$$x^2e^x(\underbrace{A-2A+A}_{0}) + xe^x(\underbrace{4A-4A}_{0}) + e^x(2A) = -4e^x$$

$$2Ae^x = -4e^x$$

$$2A = -4 \Rightarrow A = -2$$

So $y_p = -2x^2e^x$. and the general solution

$$\boxed{y = c_1 e^x + c_2 x e^x - 2x^2 e^x}$$