September 24 Math 3260 sec. 53 Fall 2025

Chapter 3: Matrix Algebra

For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ and scalar c, we defined

- the row and column vectors,

$$\mathrm{Row}_i(A) = \langle a_{i1}, a_{i2}, \dots, a_{in} \rangle, \quad \text{for} \quad i = 1, \dots, m$$

$$\text{and} \quad \mathrm{Col}_j(A) = \langle a_{1j}, a_{2j}, \dots, a_{mj} \rangle, \quad \text{for} \quad i = j, \dots, n$$

- matrix addition: $A + B = [a_{ij} + b_{ij}]$, and
- scalar multiplication $cA = [ca_{ij}]$.

Recall that the entries can be referenced using two notations,

$$a_{ij} = A_{(i,j)}.$$

3.3 Multiplication of Two Matrices

Suppose A is an $m \times p$ matrix and B is a $p \times n$ matrix. Then the product AB is the $m \times n$ matrix

$$AB = \left[(AB)_{(i,j)}\right], \quad \text{where} \quad (AB)_{(i,j)} = \mathrm{Row}_i(A) \cdot \mathrm{Col}_j(B).$$

If the number of columns of A does not match the number of rows of B, then AB is not defined.

Transpose

Suppose $A = [a_{ij}]$ is an $m \times n$ matrix. The matrix A^T, called the **transpose** of A, is the $n \times m$ matrix defined by

$$\left(A^T\right)_{(i,j)} = A_{(j,i)}.$$

$$\mathrm{Row}_i(A) = \mathrm{Col}_i(A^T) \quad \text{and} \quad \mathrm{Col}_j(A) = \mathrm{Row}_j(A^T).$$

Algebraic Properties

Let A, B, and C be matrices such that the appropriate sums and products are defined, and let r be a scalar. Then

(i) $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$

(ii) $A(rB) = (rA)B = r(AB)$

(iii) $(A^T)^T = A$

(iv) $(A+B)^T = A^T + B^T$

(v) $(rA)^T = rA^T$

(vi) $(AB)^T = B^T A^T$

Example: $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 3 \end{bmatrix}$, $\quad B = \begin{bmatrix} -2 & 1 \\ 2 & -4 \end{bmatrix}$

Evaluate

1. $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 3 \end{bmatrix}$

2. $AB = \begin{bmatrix} -2 & 1 \\ -6 & 6 \\ 0 & -9 \end{bmatrix}$

   $3\times 2,\ 2\times 2$

   $\downarrow$

   3×2

3. $BA^T = \begin{bmatrix} -2 & -5 & -3 \\ 2 & 8 & -6 \end{bmatrix}$

   $2\times 2 \quad 2\times 3$

   ✓

   $\downarrow$

   2×3

3.5 Multiplication of a Vector by a Matrix

The Product $A\vec{x}$

Suppose A is an $m \times n$ matrix and let $\vec{x}$ be a vector in R^n. Then the **matrix-vector product** $A\vec{x}$ is the vector in R^m given by

$$A\vec{x} = \langle \mathrm{Row}_1(A) \cdot \vec{x}, \mathrm{Row}_2(A) \cdot \vec{x}, \ldots, \mathrm{Row}_m(A) \cdot \vec{x} \rangle \,.$$

- For $A\vec{x}$ to be defined, $\vec{x}$ must have the same number of entries as A has columns.
- The vector $A\vec{x}$ has the same number of entries as A has rows.

$$\underbrace{\underset{m \times n}{A}\ \underset{R^n}{\vec{x}}}_{R^m}$$

Example

Find the product $A\vec{x}$ if $A = \begin{bmatrix} 3 & 0 & 1 & 3 \\ 1 & -1 & 2 & 0 \\ 0 & 2 & 0 & -1 \end{bmatrix}$ and $\vec{x} = \langle 1, -3, 0, 2 \rangle$.

A $\vec{x}$

3×4, $\mathbb{R}^4$

$\downarrow$

$\mathbb{R}^3$

$$A\vec{x} = \langle \mathrm{Row}_1(A)\cdot\vec{x}, \mathrm{Row}_2(A)\cdot\vec{x}, \mathrm{Row}_3(A)\cdot\vec{x} \rangle$$

$$\mathrm{Row}_1(A)\cdot\vec{x} = \langle 3, 0, 1, 3\rangle \cdot \langle 1, -3, 0, 2\rangle = 3+0+0+6 = 9$$

$$\mathrm{Row}_2(A)\cdot\vec{x} = \langle 1, -1, 2, 0\rangle \cdot \langle 1, -3, 0, 2\rangle = 1+3+0+0 = 4$$

$$\mathrm{Row}_3(A)\cdot\vec{x} = \langle 0, 2, 0, -1\rangle \cdot \langle 1, -3, 0, 2\rangle = 0-6+0-2 = -8$$

$$A\vec{x} = \langle 9, 4, -8 \rangle$$

Compatibility w/ Matrix Multiplication

If $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$ is a vector in R^n, and we define an $n \times 1$ matrix X via $\mathrm{Col}_1(X) = \vec{x}$, i.e.,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$\underset{m\times n}{A}\ \underset{n\times 1}{X} \downarrow m\times 1$

then the matrix-vector product

$$AX = \begin{bmatrix} \mathrm{Row}_1(A) \cdot \mathrm{Col}_1(X) \\ \mathrm{Row}_2(A) \cdot \mathrm{Col}_1(X) \\ \vdots \\ \mathrm{Row}_m(A) \cdot \mathrm{Col}_1(X) \end{bmatrix} = \begin{bmatrix} \mathrm{Row}_1(A) \cdot \vec{x} \\ \mathrm{Row}_2(A) \cdot \vec{x} \\ \vdots \\ \mathrm{Row}_m(A) \cdot \vec{x} \end{bmatrix}$$

is the $m \times 1$ matrix such that $\mathrm{Col}_1(AX) = A\vec{x}$.

Alternate Formulation of $A\vec{x}$

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{and} \quad \vec{x} = \langle x_1, x_2, x_3 \rangle.$$

Evaluate the product $A\vec{x}$ and find three vectors $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ such that $A\vec{x}$ is a linear combination $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3$.

$A\vec{x}$

2×3 $\mathbb{R}^3$

$\downarrow$

$\mathbb{R}^2$

$$A\vec{x} = \langle \text{Row}_1(A)\cdot\vec{x},\ \text{Row}_2(A)\cdot\vec{x} \rangle$$

$$= \langle a_{11}x_1 + a_{12}x_2 + a_{13}x_3,\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \rangle$$

$$= \langle a_{11}x_1, a_{21}x_1 \rangle + \langle a_{12}x_2, a_{22}x_2 \rangle + \langle a_{13}x_3, a_{23}x_3 \rangle$$

$$= x_1\langle a_{11}, a_{21} \rangle + x_2\langle a_{12}, a_{22} \rangle + x_3\langle a_{13}, a_{23} \rangle$$

$$\vec{u}_1 = \langle a_{11}, a_{21} \rangle$$

$$\vec{u}_2 = \langle a_{12}, a_{22} \rangle$$

$$\vec{u}_3 = \langle a_{13}, a_{23} \rangle$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$\vec{u}_i = \mathrm{Col}_i(A)$$

Alternate Formulation of $A\vec{x}$

For $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in R^n and A and $m \times n$ matrix,

$$A\vec{x} = x_1 \operatorname{Col}_1(A) + x_2 \operatorname{Col}_2(A) + \cdots + x_n \operatorname{Col}_n(A).$$

That is, $A\vec{x}$ is the vector in R^m that is the linear combination of the columns of A with the entries of $\vec{x}$ as the weights.

Remark: We have two equivalent definitions for the product $A\vec{x}$. One focuses on the role of the rows of A, and the other focuses on the role of the columns of A.

Example

Find the product $A\vec{x}$ using the second formulation, where

$$A = \begin{bmatrix} 3 & 0 & 1 & 3 \\ 1 & -1 & 2 & 0 \\ 0 & 2 & 0 & -1 \end{bmatrix} \text{ and } \vec{x} = \langle 1, -3, 0, 2 \rangle.$$

$$A\vec{x} = x_1 \, \mathrm{Col}_1(A) + x_2 \, \mathrm{Col}_2(A) + x_3 \, \mathrm{Col}_3(A) + x_4 \, \mathrm{Col}_4(A)$$

$$= 1\langle 3,1,0\rangle + (-3)\langle 0,-1,2\rangle + 0\langle 1,2,0\rangle + 2\langle 3,0,-1\rangle$$

$$= \langle 3,1,0\rangle + \langle 0,3,-6\rangle + \langle 0,0,0\rangle + \langle 6,0,-2\rangle$$

$$= \langle 3+0+0+6,\ 1+3+0+0,\ 0-6-0-2\rangle$$

$$= \langle 9,4,-8\rangle$$

Revisiting the Product AB

$$AB = \begin{bmatrix} \text{Row}_1(A)\cdot\text{Col}_1(B) & \text{Row}_1(A)\cdot\text{Col}_2(B) & \cdots & \text{Row}_1(A)\cdot\text{Col}_n(B) \\ \text{Row}_2(A)\cdot\text{Col}_1(B) & \text{Row}_2(A)\cdot\text{Col}_2(B) & \cdots & \text{Row}_2(A)\cdot\text{Col}_n(B) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Row}_m(A)\cdot\text{Col}_1(B) & \text{Row}_m(A)\cdot\text{Col}_2(B) & \cdots & \text{Row}_m(A)\cdot\text{Col}_n(B) \end{bmatrix}$$

Look at the first column:

$$\begin{bmatrix} \text{Row}_1(A)\cdot\text{Col}_1(B) \\ \text{Row}_2(A)\cdot\text{Col}_1(B) \\ \vdots \\ \text{Row}_m(A)\cdot\text{Col}_1(B) \end{bmatrix}$$

Revisiting the Product AB

$$AB = \begin{bmatrix} \mathrm{Row}_1(A)\cdot\mathrm{Col}_1(B) & \mathrm{Row}_1(A)\cdot\mathrm{Col}_2(B) & \cdots & \mathrm{Row}_1(A)\cdot\mathrm{Col}_n(B) \\ \mathrm{Row}_2(A)\cdot\mathrm{Col}_1(B) & \mathrm{Row}_2(A)\cdot\mathrm{Col}_2(B) & \cdots & \mathrm{Row}_2(A)\cdot\mathrm{Col}_n(B) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A)\cdot\mathrm{Col}_1(B) & \mathrm{Row}_m(A)\cdot\mathrm{Col}_2(B) & \cdots & \mathrm{Row}_m(A)\cdot\mathrm{Col}_n(B) \end{bmatrix}$$

Look at the first column:

$$\begin{bmatrix} \mathrm{Row}_1(A)\cdot\mathrm{Col}_1(B) \\ \mathrm{Row}_2(A)\cdot\mathrm{Col}_1(B) \\ \vdots \\ \mathrm{Row}_m(A)\cdot\mathrm{Col}_1(B) \end{bmatrix} \quad \text{looks like} \quad \begin{bmatrix} \mathrm{Row}_1(A)\cdot\mathrm{Col}_1(X) \\ \mathrm{Row}_2(A)\cdot\mathrm{Col}_1(X) \\ \vdots \\ \mathrm{Row}_m(A)\cdot\mathrm{Col}_1(X) \end{bmatrix}$$

Revisiting the Product AB

$$AB = \begin{bmatrix} \text{Row}_1(A)\cdot\text{Col}_1(B) & \text{Row}_1(A)\cdot\text{Col}_2(B) & \cdots & \text{Row}_1(A)\cdot\text{Col}_n(B) \\ \text{Row}_2(A)\cdot\text{Col}_1(B) & \text{Row}_2(A)\cdot\text{Col}_2(B) & \cdots & \text{Row}_2(A)\cdot\text{Col}_n(B) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Row}_m(A)\cdot\text{Col}_1(B) & \text{Row}_m(A)\cdot\text{Col}_2(B) & \cdots & \text{Row}_m(A)\cdot\text{Col}_n(B) \end{bmatrix}$$

Look at the first column:

$$\begin{bmatrix} \text{Row}_1(A)\cdot\text{Col}_1(B) \\ \text{Row}_2(A)\cdot\text{Col}_1(B) \\ \vdots \\ \text{Row}_m(A)\cdot\text{Col}_1(B) \end{bmatrix} \quad \text{looks like} \quad \begin{bmatrix} \text{Row}_1(A)\cdot\text{Col}_1(X) \\ \text{Row}_2(A)\cdot\text{Col}_1(X) \\ \vdots \\ \text{Row}_m(A)\cdot\text{Col}_1(X) \end{bmatrix}$$

This is the matrix-vector product of A with the vector $\text{Col}_1(B)$

$$\text{Col}_1(AB) = A\,\text{Col}_1(B).$$

Revisiting the Product AB

Column Vectors of the Product AB

If A in an $m \times p$ matrix and B is a $p \times n$ matrix, then the product AB is the $m \times n$ matrix whose columns are defined via the matrix-vector products

$$\mathrm{Col}_i(AB) = A\,\mathrm{Col}_i(B).$$

This says that each column vector of the product AB is a matrix-vector product that looks like "$A\vec{x}$" where the matrix is the matrix A from the left and the vector is a column vector from the matrix on the right, B.

Example

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Determine the product AB by computing the column vectors as the matrix-vector products $\mathrm{Col}_i(AB) = A\,\mathrm{Col}_i(B)$.

AB

$2\times 2,\ 2\times 3$

$\Downarrow$

2×3

$$\mathrm{Col}_1(AB) = A\,\mathrm{Col}_1(B)$$

$$= \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \langle 2, 1\rangle$$

$$= \langle 2-3, -4+2\rangle = \langle -1, -2\rangle$$

$$\mathrm{Col}_2(AB) = A\,\mathrm{Col}_2(B) = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \langle 0, -4\rangle$$

$$= \langle 0+12, 0-8\rangle = \langle 12, -8\rangle$$

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

$$\mathrm{Col}_3(AB) = A\,\mathrm{Col}_3(B)$$

$$= \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \langle 2, 6 \rangle = \langle 2-18, -4+12 \rangle = \langle -16, 8 \rangle$$

$$AB = \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$$

The Product AB

If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then the product AB is the $m \times n$ matrix whose row vectors are given by

$$\mathrm{Row}_i(AB) = B^T\,\mathrm{Row}_i(A), \quad i = 1, \ldots, m.$$

We know that $(XY)^T = Y^T X^T$ and $\mathrm{Col}_i(XY) = X\,\mathrm{Col}_i(Y)$.

$$\mathrm{Row}_i(AB) = \mathrm{Col}_i\left((AB)^T\right)$$

$$= \mathrm{Col}_i\left(B^T A^T\right)$$

$$= B^T\,\mathrm{Col}_i\left(A^T\right) = B^T\,\mathrm{Row}_i(A)$$

The Product $A^T\vec{x}$

Suppose A is an $m\times n$ matrix and $\vec{x}$ is a vector in R^m. Then the product $A^T\vec{x}$ is the vector in R^n given by

$$\begin{aligned} A^T\vec{x} &= \langle \mathrm{Col}_1(A)\cdot\vec{x}, \mathrm{Col}_2(A)\cdot\vec{x}, \dots, \mathrm{Col}_n(A)\cdot\vec{x}\rangle \\ &= x_1\,\mathrm{Row}_1(A) + x_2\,\mathrm{Row}_2(A) + \cdots + x_m\,\mathrm{Row}_m(A). \end{aligned}$$

Since $\mathrm{Row}_i(A^T) = \mathrm{Col}_i(A)$ and $\mathrm{Col}_i(A^T) = \mathrm{Row}_i(A)$, these match our previous representations for A^T instead of A:

$$A^T\vec{x} = \left\langle \underbrace{\mathrm{Row}_1(A^T)}_{\mathrm{Col}_1(A)}\cdot\vec{x}, \underbrace{\mathrm{Row}_2(A^T)}_{\mathrm{Col}_2(A)}\cdot\vec{x}, \dots, \underbrace{\mathrm{Row}_n(A^T)}_{\mathrm{Col}_n(A)}\cdot\vec{x}\right\rangle$$

$$A^T\vec{x} = x_1\underbrace{\mathrm{Col}_1(A^T)}_{\mathrm{Row}_1(A)} + x_2\underbrace{\mathrm{Col}_1(A^T)}_{\mathrm{Row}_2(A)} + \cdots + x_m\underbrace{\mathrm{Col}_m(A^T)}_{\mathrm{Row}_m(A)}$$

Example

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix}$, $\vec{x} = \langle 3, 1 \rangle$ and $\vec{y} = \langle -2, -1, 1 \rangle$. Evaluate

$$A\vec{x}, \quad A^T\vec{y}, \quad (A\vec{x}) \cdot \vec{y}, \quad \text{and} \quad \vec{x} \cdot (A^T\vec{y}).$$

$$A\vec{x} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix} \langle 3, 1 \rangle = \langle 3-1, 0+2, -6+1 \rangle$$

$$= \langle 2, 2, -5 \rangle$$

$$A^T\vec{y} = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 2 & 1 \end{bmatrix} \langle -2, -1, 1 \rangle = \langle -2+0-2, 2-2+1 \rangle$$

$$= \langle -4, 1 \rangle$$

$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix}$, $\vec{x} = \langle 3, 1\rangle$ and $\vec{y} = \langle -2, -1, 1\rangle$

$A\vec{x} = \langle 2, 2, -5\rangle$, $A^T\vec{y} = \langle -4, 1\rangle$

$(A\vec{x})\cdot\vec{y} = \langle 2, 2, -5\rangle \cdot \langle -2, -1, 1\rangle$

$= -4 - 2 - 5 \quad = -11$

$\vec{x}\cdot(A^T\vec{y}) = \langle 3, 1\rangle \cdot \langle -4, 1\rangle = -12 + 1 = -11$

We found that

$$(A\vec{x})\cdot\vec{y} = \vec{x}\cdot(A^T\vec{y})$$

Dot Products & Transposes

Suppose A is an $m \times n$ matrix, $\vec{x}$ is a vector in R^n and $\vec{y}$ is a vector in R^m. Then

$$(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot \left(A^T \vec{y}\right).$$

Note that the left side of the equation is a dot product of vectors in R^m, whereas the right side is a dot product in R^n. So the number of products and sums on each side is different when $m \neq n$, but the final value is the same.