September 25 Math 2306 sec. 51 Spring 2023

Consider the second order homogeneous ODE

$$x^2y'' - xy' + y = 0 \quad \text{for} \quad x > 0.$$

- Note that $y_1 = x$ is a solution.
- **Question:** Is $y = c_1 y_1$ the general solution? (Why/why not?)

No, the ODE is 2^{nd} order, so the solution has 2 lin. independent parts.

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Standard Form

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Some things to keep in mind:

- Every fundamental solution set has two linearly independent solutions y_1 and y_2,
- The general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we know one solution $y_1(x)$. This section is about a process called **Reduction of order**. Reduction of order is a method for finding a second solution by assuming that

$$y_2(x) = u(x)y_1(x).$$

The goal is to find the unknown function u.

Context

- We start with a second order, linear, homogeneous ODE in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

- We know one solution y_1. (Keep in mind that y_1 is a known!)
- We know there is a second linearly independent solution (section 6 theory says so).
- We try to find y_2 by guessing that it can be found in the form

$$y_2(x) = u(x)y_1(x)$$

where the goal becomes finding u.

- **Due to linear independence, we know that u cannot be constant.**

Example

Find the general solution to the ODE $x^2y'' - xy' + y = 0$ for $x > 0$ given that $y_1(x) = x$ is one solution.

The ODE in standard form is

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0 \qquad y_1 = x$$

Set $\quad y_2 = uy_1 = xu \qquad$ sub this into the ODE

$$y_2 = xu$$
$$y_2' = 1u + xu' = u + xu'$$
$$y_2'' = u' + 1u' + xu'' = 2u' + xu''$$

$$y_2'' - \frac{1}{x}y_2' + \frac{1}{x^2}y_2 = 0$$

$$2u' + xu'' - \frac{1}{x}(u + xu') + \frac{1}{x^2}(xu) = 0$$

$$2u' + xu'' - \cancel{\tfrac{1}{x}u} - u' + \cancel{\tfrac{1}{x}u} = 0$$

$$xu'' + u' = 0.$$

Let $w = u'$, then $w' = u''$. * we know $w \neq 0$

Then w solves the 1st order equation

$$xw' + w = 0$$

this is both linear and separable!

Let's separate variables

$$x\frac{dw}{dx} = -w \Rightarrow \frac{dw}{dx} = \frac{-w}{x}$$

$$\frac{1}{w}\,dw = \frac{-1}{x}\,dx$$

$$\int \frac{1}{w}\,dw = -\int \frac{1}{x}\,dx$$

$$\ln|w| = -\ln x + C$$

$$|w| = e^{-\ln x + C} = e^{C} \cdot e^{-\ln x} = e^{C} x^{-1}$$

Let $k = \pm e^{C}$, $w = k x^{-1} = \frac{k}{x}$

$$w = u' \Rightarrow u = \int w \, dx = \int \frac{k}{x} dx = k \ln x + c_1$$

Now, $y_2 = u y_1 = (k \ln x + c_1) x = k x \ln x + c_1 x$

The general solution $y = c_1 y_1 + c_2 y_2$

We can take $y_2 = x \ln x$ (e.g., $k = 1$, $c_1 = 0$)

$$\boxed{\text{The general solution is} \quad y = c_1 x + c_2 x \ln x}$$

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - \text{is known.}$$

Let $y_2 = uy_1$ sub

$y_2' = u'y_1 + uy_1'$

$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$

$= u''y_1 + 2u'y_1' + uy_1''$

$y_2'' + P(x)y_2' + Q(x)y_2 = 0$

We know that $y_1'' + P(x)y_1' + Q(x)y_1 = 0.$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)\left(u'y_1 + uy_1'\right) + Q(x)\left(uy_1\right) = 0$$

Collect u'', u', u

$$y_1 u'' + \left(2y_1' + P(x)y_1\right)u' + \underbrace{\left(y_1'' + P(x)y_1' + Q(x)y_1\right)}_{0 \text{ since } y_1 \text{ is a solution}} u = 0$$

we have

$$y_1 u'' + \left(2y_1' + P(x)y_1\right)u' = 0$$

Letting $w = u'$, $w' = u''$ and w solves the 1st order ODE

$$y_1 w' + \left(2y_1' + P(x)y_1\right)w = 0$$

both separable and linear.

Separating variables

$$y_1 \frac{dw}{dx} = -\left(2\frac{dy_1}{dx} + P(x)\,y_1\right) w$$

$$\frac{1}{w}\frac{dw}{dx} = -\left(2\frac{\frac{dy_1}{dx}}{y_1} + P(x)\right)$$

$$\frac{1}{w}\,dw = -2\frac{\frac{dy_1}{dx}}{y_1}\,dx - P(x)\,dx$$

$$\frac{1}{w}\,dw = -2\frac{dy_1}{y_1} - P(x)\,dx$$

$$\int \frac{1}{w}\,dw = -2\int \frac{dy_1}{y_1} - \int P(x)\,dx$$

$$\ln|w| = -2\ln|y_1| - \int P(x)\,dx + C$$

$$e^{\ln|w|} = e^{\ln y_1^{-2} - \int P(x)\,dx + C}$$

$$|w| = y_1^{-2} \cdot e^{-\int P(x)\,dx} \cdot e^{C}$$

Letting $e^c = \pm 1$, we discard the absolute value bars to get

$$w = \frac{e^{-\int p(x)dx}}{y_1^2}$$

Since $w = u'$, $u = \int w \, dx$

$$u = \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

$$y_2 = u y_1$$

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2(x) = y_1(x)u(x) \quad \text{where} \quad u(x) = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 1$$

The second solution $y_2 = uy_1$ where

$$u = \int \frac{e^{-\int P(x)dx}}{y_1^2}\, dx.$$

Here $P(x) = 4$, $-\int P(x)\,dx = -\int 4\,dx = -4x$

$$(y_1)^2 = (e^{-2x})^2 = e^{-4x}$$

$$u = \int \frac{e^{-4x}}{e^{-4x}}\, dx = \int dx = x$$

$y_2 = u y_1 = x e^{-2x}$

The general solution to the ODE is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

Apply $y(0) = 1$ and $y'(0) = 1$

$$y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = 1 \Rightarrow c_1 = 1$$

$$y'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 \cdot 0 \cdot e^0 = 1$$

$$-2c_1 + c_2 = 1 \Rightarrow c_2 = 1 + 2c_1 = 1 + 2 = 3$$

The solution to the IVP is

$$y = e^{-2x} + 3x e^{-2x}$$