September 25 Math 2306 sec. 52 Spring 2023

Consider the second order homogeneous ODE

$$x^2y'' - xy' + y = 0 \quad \text{for} \quad x > 0.$$

- Note that $y_1 = x$ is a solution.
- **Question:** Is $y = c_1y_1$ the general solution? (Why/why not?)

No, the ODE is 2^{nd} order, there should be a c_2y_2 part.

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Standard Form

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Some things to keep in mind:

- Every fundamental solution set has two linearly independent solutions y_1 and y_2,
- The general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we know one solution $y_1(x)$. This section is about a process called **Reduction of order**. Reduction of order is a method for finding a second solution by assuming that

$$y_2(x) = u(x) y_1(x).$$

The goal is to find the unknown function u.

Context

- We start with a second order, linear, homogeneous ODE in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

- We know one solution y_1. (Keep in mind that y_1 is a known!)
- We know there is a second linearly independent solution (section 6 theory says so).
- We try to find y_2 by guessing that it can be found in the form

$$y_2(x) = u(x)y_1(x)$$

  where the goal becomes finding u.
- **Due to linear independence, we know that u <u>cannot</u> be constant.**

Example

Find the general solution to the ODE $x^2y'' - xy' + y = 0$ for $x > 0$ given that $y_1(x) = x$ is one solution.

In standard form, the ODE is

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0 \qquad y_1 = x$$

Set $y_2 = uy_1 = xu$ sub into the ODE

$$y_2' = 1u + xu' = u + xu'$$

$$y_2'' = u' + 1u' + xu'' = 2u' + xu''$$

$$y_2'' - \frac{1}{x}y_2' + \frac{1}{x^2}y_2 = 0$$

$$2u' + xu'' - \frac{1}{x}(u + xu') + \frac{1}{x^2}(xu) = 0$$

$$2u' + xu'' - \cancel{\frac{1}{x}u} - u' + \cancel{\frac{1}{x}u} = 0$$

$$xu'' + u' = 0$$

Let $w = u'$, then $w' = u''$

* $w \neq 0$ since u is not constant

w solves the 1st order ODE

$$xw' + w = 0$$

This is linear and separable. Let's separate variables

$$x\frac{dw}{dx} + w = 0 \Rightarrow x\frac{dw}{dx} = -w \Rightarrow \frac{dw}{dx} = \frac{-w}{x}$$

$$\frac{1}{w}\,dw = -\frac{1}{x}\,dx$$

$$\int \frac{1}{w}\,dw = \int \frac{-1}{x}\,dx$$

$$\ln|w| = -\ln x + c$$

$$e^{\ln|w|} = e^{\ln x^{-1} + c} = e^{c} e^{\ln x^{-1}}$$

$$|w| = e^{c} x^{-1} \qquad \text{Let} \quad k = \pm e^{c}$$

$$w = k x^{-1}$$

Since $w = u'$, $u = \int w \, dx$

$$u = \int k x^{-1} \, dx = k \ln x + k_1$$

$y_1 = x$

$$y_2 = u y_1 = (k \ln x + k_1) x = k x \ln x + k_1 x$$

The general solution

$$y = c_1 y_1 + c_2 y_2$$

we can take $y_2 = x\ln x$

$$\boxed{y = c_1 x + c_2 x \ln x}$$

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - \text{is known.}$$

Set $y_2 = uy_1$ sub into the ODE

$y_2' = u'y_1 + uy_1'$

$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$

$= u''y_1 + 2u'y_1' + uy_1''$

$y_2'' + P(x)y_2' + Q(x)y_2 = 0$

We know that $\quad y_1'' + P(x)y_1' + Q(x)y_1 = 0.$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)\left(u'y_1 + uy_1'\right) + Q(x)(uy_1) = 0$$

Collect u'', u', and u.

$$y_1 u'' + \left(2y_1' + P(x)y_1\right)u' + \underbrace{\left(y_1'' + P(x)y_1' + Q(x)y_1\right)}_{\overset{\shortparallel}{0}}u = 0$$

since y_1 is a solution

The ODE for u is

$$y_1 u'' + \left(2y_1' + P(x)y_1\right)u' = 0$$

If $w = u'$, then w satisfies

$$y_1 w' + \left(2y_1' + P(x)y_1\right)w = 0$$

A first order linear and separable ODE.

Let's separate.

$$y_1 \frac{dw}{dx} + \left(2 \frac{dy_1}{dx} + P(x) y_1\right) w = 0$$

$$y_1 \frac{dw}{dx} = -\left(2 \frac{dy_1}{dx} + P(x) y_1\right) w$$

$$\frac{1}{w} \frac{dw}{dx} = -\left(2 \frac{\frac{dy_1}{dx}}{y_1} + P(x) \frac{y_1}{y_1}\right)$$

$$\frac{1}{w} \frac{dw}{dx} dx = \left(-2 \frac{\frac{dy_1}{dx}}{y_1} - P(x)\right) dx$$

$$\int \frac{1}{w} dw = \int -2 \frac{dy_1}{y_1} - \int P(x)\, dx$$

$$\ln|w| = -2 \ln|y_1| - \int P(x)\, dx + C$$

$$e^{\ln|w|} = e^{-2\ln|y_1| - \int P(x) dx + C}$$

$$|w| = e^{\ln y_1^{-2}} \cdot e^{-\int P(x)dx} \cdot e^{c}$$

Let e^c be replaced with ± 1 to discard the absolute value bars on w.

$$w = y_1^{-2} e^{-\int P(x)dx} = \frac{e^{-\int P(x)\,dx}}{y_1^2}$$

$$u = \int w\,dx \Rightarrow u = \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx$$

and $y_2 = u y_1$

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2(x) = y_1(x)u(x) \quad \text{where} \quad u(x) = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 1$$

Find y_2 using reduction of order,

$y_2 = u y_1$ where $u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$

$P(x) = 4$, $-\int P(x)\,dx = -\int 4\,dx = -4x$

$e^{-\int P(x)\,dx} = e^{-4x}$

$(y_1)^2 = \left(e^{-2x}\right)^2 = e^{-4x}$

$$u = \int \frac{e^{-\int p(x)dx}}{(y_1)^2}\,dx = \int \frac{e^{-4x}}{e^{-4x}}\,dx = \int dx = x$$

So $\quad y_2 = u y_1 = x e^{-2x}$

The general solution

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 e^{-2x} + C_2 x e^{-2x}$$

Apply $\quad y(0) = 1 \quad$ and $\quad y'(0) = 1$

$$y' = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x}$$

$$y(0) = C_1 e^{0} + C_2 \cdot 0 \cdot e^{0} = 1 \quad \Rightarrow \quad C_1 = 1$$

$$y'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 \cdot 0 \cdot e^0 = 1$$

$$-2c_1 + c_2 = 1 \Rightarrow c_2 = 1 + 2c_1 = 1 + 2 = 3$$

The solution to the IVP is

$$\boxed{y = e^{-2x} + 3xe^{-2x}}$$