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Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0, \quad \text{with } a\neq 0.$$

The **characteristic equation** for this ODE is the second degree polynomial equation

$$am^2+bm+c=0.$$

If m is a solution to this polynomial equation, then $y=e^{mx}$ is a solution to the differential equation. There are three cases.

[1] We'll extend the result to higher order at the end of this section.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 xe^{mx}.$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

Let α be the real part of the complex roots and β be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Multiple Choice

We can use a characteristic equation to find the **general solution** of which ODE(s)?

1. $y'' + xy' - y = 0$
2. $y'' + 2y' + y = 0$
3. $y'' + 9y = 0$
4. $y'' + 9y = \sqrt{x}$
5. all of the above
6. 1., 2., and 3. only
7. 2., and 3. only
8. 2., 3., and 4. only

Example

Solve the IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

Characteristic eqn: $m^2 + 6m + 9 = 0$

$(m+3)^2 = 0 \Rightarrow m = -3$ double root

The solutions are

$y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$

The general solution is

$y = c_1 e^{-3x} + c_2 x e^{-3x}$

Apply $y(0)=4$ and $y'(0)=0$.

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = 4 \implies c_1 = 4$$

$$y'(0) = -3c_1 e^0 + c_2 e^0 - 3c_2 \cdot 0 \cdot e^0 = 0 \quad -3c_1 + c_2 = 0$$

$$\implies c_2 = 3c_1 = 3(4) = 12$$

$$\boxed{\begin{array}{l}\text{The solution to the IVP is} \\ y = 4e^{-3x} + 12x e^{-3x}\end{array}}$$

Example

Find the general solution.

$$y'' + 4y = 0$$

Characteristic eqn : $m^2 + 4 = 0$

$\Longrightarrow \quad m^2 = -4$

$m = \pm\sqrt{-4} = \pm 2i$

$m = \alpha \pm i\beta$ with $\alpha = 0$ and $\beta = 2$

The solutions are

$y_1 = e^{0x} \cos(2x)$, $y_2 = e^{0x} \sin(2x)$

The general solution is

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Exercise

Determine the general solution of the second order, linear, constant coefficient ODE

$$y'' + 3y' + 2y = 0$$

$y = c_1 e^{-2x} + c_2 e^{-x}$ or $y = c_1 e^{-x} + c_2 e^{-2x}$

$m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m_1 = -2$

$m_2 = -1$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.

- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ for each pair of complex roots.

- It may require a computer algebra system to find the roots for a high degree polynomial.

Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, m may be a root of multiplicity k where $1 \le k \le n$.

- If a real root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2e^{mx}, \quad \ldots, \quad x^{k-1}e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x),\ldots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

Example

Find the general solution of the ODE.

$y''' - 3y'' + 3y' - y = 0$

Linear, homogeneous, constant coef.

The characteristic equation is

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$\Longrightarrow \quad (m-1)^3 = 0 \quad \Longrightarrow m = 1$$ is a triple root

The solutions are

$$y_1 = e^{1x}, \quad y_2 = x e^x, \quad y_3 = x^2 e^x$$

The general solution is

$$y = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$$

Example

Consider the 7^{th} order homogeneous ODE

$$y^{(7)} - 10y^{(6)} + 48y^{(5)} - 144y^{(4)} + 288y''' - 384y'' + 320y' - 128y = 0$$

The characteristic equation, completely factored, is

constant coef.

$$((m-1)^2 + 3)^2(m-2)^3 = 0.$$

Find the general solution.

A fundamental solution set must have 7 lin. independent solutions.

Find the roots:

$(m-2)^3 = 0 \implies m = 2$ triple root

$y_1 = e^{2x}$, $y_2 = xe^{2x}$, $y_3 = x^2e^{2x}$

$$\left((m-1)^2+3\right)^2=0 \qquad \left((m-1)^2+3\right)\left((m-1)^2+3\right)=0$$

$$(m-1)^2+3=0 \implies (m-1)^2=-3$$

$$\implies (m-1)=\pm\sqrt{-3} \implies m=1\pm\sqrt{3}\,i$$

$m=1\pm\sqrt{3}\,i$ are both double roots

$$y_4=e^{1x}\cos(\sqrt{3}\,x), \qquad y_5=e^{1x}\sin(\sqrt{3}\,x)$$

$$y_6=x e^{x}\cos(\sqrt{3}x), \qquad y_7=x e^{x}\sin(\sqrt{3}\,x)$$

The general solution

$$y = C_1 e^{2x} + C_2 x e^{2x} + C_3 x^2 e^{2x} + C_4 e^{x} \cos(\sqrt{3}x)$$

$$+ C_5 e^{x} \sin(\sqrt{3}x) + C_6 x e^{x} \cos(\sqrt{3}x) + C_7 x e^{x} \sin(\sqrt{3}x)$$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e.g., e^{mx} m - constant
- sines and/or cosines, $\sin(kx)$, or $\cos(kx)$ k - constant
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example[2]

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

The left is constant coef. and the right is a polynomial.

$g(x) = 8x + 1$ is a 1st degree polynomial

Guess y_p is also a 1st degree polynomial.

Set $y_p = Ax + B$ for A, B constant

[2] We're only ignoring the y_c part to illustrate the process.

Substitute $y_p' = A, \quad y_p'' = 0$

into

$$y_p'' - 4y_p' + 4y_p = 8x+1$$

$$0 - 4(A) + 4(Ax+B) = 8x+1$$

$$4Ax + (-4A+4B) = 8x+1$$

Match like terms

$$4A = 8 \implies A = 2$$

$$-4A+4B = 1 \implies B = \frac{1}{4}(1+4A) = \frac{1}{4}(1+8)$$
$$= \frac{9}{4}$$

Hence $y_p = 2x + \frac{9}{4}$