September 26 Math 3260 sec. 51 Fall 2025

3.6 Standard Unit Vectors & Identity Matrices

Standard Unit Vectors

Let $\vec{e}_i$ denote the vector in R^n whose i^{th} entry is 1 and having all other entries zero. We will call the n such vectors in R^n the **standard unit vectors** in R^n.

There are two such vectors in R^2

$$\vec{e}_1 = \langle 1, 0\rangle, \quad \text{and} \quad \vec{e}_2 = \langle 0, 1\rangle.$$

There are three such vectors in R^3

$$\vec{e}_1 = \langle 1, 0, 0\rangle, \quad \vec{e}_2 = \langle 0, 1, 0\rangle, \quad \text{and} \quad \vec{e}_3 = \langle 0, 0, 1\rangle.$$

And so forth...

Example

Let $\vec{x} = \langle 2, -7, 12 \rangle$. Evaluate each of the dot products

1. $\vec{x} \cdot \vec{e}_1 = \langle 2, -7, 12 \rangle \cdot \langle 1, 0, 0 \rangle$
   $= 2(1) + (-7)(0) + 12(0) = 2$

2. $\vec{x} \cdot \vec{e}_2 = \langle 2, -7, 12 \rangle \cdot \langle 0, 1, 0 \rangle$
   $= 2(0) + (-7)(1) + 12(0) = -7$

3. $\vec{x} \cdot \vec{e}_3 = 12$

Matrix-Vector Products

In general, if $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$, then

$$\vec{x} \cdot \vec{e}_i = x_i, \quad \text{for each} \quad i = 1, \ldots, n.$$

What about matrices?

If A is an $m \times n$ matrix, identify the product $A\vec{e}_i$ for the i^{th} standard unit vector in R^n.

$$A\vec{e}_1 = 1\,\mathrm{Col}_1(A) + 0\,\mathrm{Col}_2(A) + \cdots + 0\,\mathrm{Col}_n(A)$$
$$= \mathrm{Col}_1(A)$$

$$A\vec{e}_2 = 0\,\mathrm{Col}_1(A) + 1\,\mathrm{Col}_2(A) + 0\,\mathrm{Col}_3(A) + \cdots + 0\,\mathrm{Col}_n(A)$$
$$= \mathrm{Col}_2(A)$$

$$A\vec{e}_i = \mathrm{Col}_i(A)$$

Identity Matrix

The $n \times n$ Identity Matrix

The $n \times n$ identity matrix, denoted I_n, is the $n \times n$ matrix defined by

$$\text{Col}_i(I_n) = \vec{e}_i.$$

That is, I_n is the $n \times n$ matrix with 1 in each diagonal entry and zero is all other entries.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Note that $\text{Col}_i(I_n) = \text{Row}_i(I_n) = \vec{e}_i$ for each $i = 1, \ldots, n$.

Matrix-Vector Product Multiplicative Identity

Recall: for $\vec{x}$ in R^n, $\vec{e}_i \cdot \vec{x} = x_i$

Let $\vec{x}$ be a vector in R^n. Evaluate the matrix-vector product

$$I_n\vec{x} = \langle \text{Row}_1(I_n)\cdot\vec{x}, \text{Row}_2(I_n)\cdot\vec{x}, \ldots, \text{Row}_n(I_n)\cdot\vec{x} \rangle$$

$$= \langle \vec{e}_1\cdot\vec{x}, \vec{e}_2\cdot\vec{x}, \ldots, \vec{e}_n\cdot\vec{x} \rangle$$

$$= \langle x_1, x_2, x_3, \ldots, x_n \rangle$$

$$= \vec{x}.$$

Matrix-Vector Product Multiplicative Identity

If $\vec{x}$ is any vector in R^n, then

$$I_n\vec{x} = \vec{x}.$$

The $n \times n$ identity matrix is the multiplicative identity for the matrix-vector product.

Matrix Multiplicative Identity

Recall: $A\vec{e}_i = \mathrm{Col}_i(A)$, $\mathrm{Col}_i(AB) = A\,\mathrm{Col}_i(B)$ and $\mathrm{Row}_i(AB) = B^T\,\mathrm{Row}_i(A)$.

$\underset{m\times n}{A}\ \underset{n\times n}{I_n}$ $\qquad$ $\underset{m\times m}{I_m}\ \underset{m\times n}{A}$

Let A be an $m \times n$ matrix. Evaluate

$\mathrm{Col}_i(AI_n) = A\,\mathrm{Col}_i(I_n) = A\vec{e}_i = \mathrm{Col}_i(A)$

$\Rightarrow AI_n = A$

and

$\mathrm{Row}_i(I_mA) = A^T\,\mathrm{Row}_i(I_m) = A^T\vec{e}_i = \mathrm{Col}_i(A^T) = \mathrm{Row}_i(A)$

$\Rightarrow I_mA = A$.

Matrix Product Multiplicative Identity

Let A be any $m \times n$ matrix. Then

$$AI_n = A, \quad \text{and} \quad I_mA = A.$$

The identity matrix (of the appropriate size) is the multiplicative identity for the product of two matrices.

The **identity** matrix is the multiplicative identity for both the matrix-vector and the matrix-matrix products. I_n is a matrix multiplication analog of the number 1.

3.7 The Associative Property of Matrix Multiplication

We know that matrix multiplication is not commutative. However, it is associative.

Associativity of Matrix Multiplication

If A is an $m \times p$ matrix, B is a $p \times q$ matrix, and C is a $q \times n$ matrix, then

$$(AB)C = A(BC).$$

Intermediate products will involve different sized matrices.

Associativity and the Matrix-Vector Product

Special Case

Suppose A is an $m \times p$ matrix and B is a $p \times n$ matrix. Let $\vec{x}$ be a vector in R^n. Then

$$(AB)\vec{x} = A\left(B\vec{x}\right).$$

Remark: This is a primary motivation for how the product AB of matrices is defined. This will become critical when we study **linear transformations** between R^m and R^n.

3.8 Matrix Equations

We'll consider two types of matrix equations.

Matrix-Vector Equation

$$A\vec{x} = \vec{y}$$

The matrix A and the vector $\vec{y}$ are known. The variable to be solved for is the vector $\vec{x}$.

Matrix-Matrix Equation

$$AX = Y$$

The matrices A and Y are known. The variable to be solved for is the matrix X.