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3.6 Standard Unit Vectors & Identity Matrices

Standard Unit Vectors

Let $\vec{e}_i$ denote the vector in R^n whose i^{th} entry is 1 and having all other entries zero. We will call the n such vectors in R^n the **standard unit vectors** in R^n.

There are two such vectors in R^2

$$\vec{e}_1 = \langle 1, 0 \rangle, \quad \text{and} \quad \vec{e}_2 = \langle 0, 1 \rangle.$$

There are three such vectors in R^3

$$\vec{e}_1 = \langle 1, 0, 0 \rangle, \quad \vec{e}_2 = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \vec{e}_3 = \langle 0, 0, 1 \rangle.$$

And so forth...

Example

Let $\vec{x} = \langle 2, -7, 12 \rangle$. Evaluate each of the dot products

1. $\vec{x} \cdot \vec{e}_1$ $= \langle 2, -7, 12 \rangle \cdot \langle 1, 0, 0 \rangle$
   $= 2(1) + (-7)(0) + 12(0) = 2$

2. $\vec{x} \cdot \vec{e}_2$ $= \langle 2, -7, 12 \rangle \cdot \langle 0, 1, 0 \rangle$
   $= 2(0) + (-7)(1) + 12(0) = -7$

3. $\vec{x} \cdot \vec{e}_3$ $= 12$

Matrix-Vector Products

In general, if $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$, then

$$\vec{x} \cdot \vec{e}_i = x_i, \quad \text{for each} \quad i = 1, \ldots, n.$$

What about matrices?

If A is an $m \times n$ matrix, identify the product $A\vec{e}_i$ for the i^{th} standard unit vector in R^n.

$$A\vec{e}_1 = 1\,\text{Col}_1(A) + 0\,\text{Col}_2(A) + \cdots + 0\,\text{Col}_n(A)$$
$$= \text{Col}_1(A)$$

$$A\vec{e}_2 = 0\,\text{Col}_1(A) + 1\,\text{Col}_2(A) + 0\,\text{Col}_3(A) + \cdots + 0\,\text{Col}_n(A)$$
$$= \text{Col}_2(A)$$

$$\vdots$$

$$A\vec{e}_i = \text{Col}_i(A).$$

Identity Matrix

The $n \times n$ Identity Matrix

The $n \times n$ identity matrix, denoted I_n, is the $n \times n$ matrix defined by

$$\mathrm{Col}_i(I_n) = \vec{e}_i.$$

That is, I_n is the $n \times n$ matrix with 1 in each diagonal entry and zero is all other entries.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Note that $\mathrm{Col}_i(I_n) = \mathrm{Row}_i(I_n) = \vec{e}_i$ for each $i = 1, \ldots, n$.

Matrix-Vector Product Multiplicative Identity

Recall: for $\vec{x}$ in R^n, $\vec{e}_i \cdot \vec{x} = x_i$

Let $\vec{x}$ be a vector in R^n. Evaluate the matrix-vector product

$$I_n\vec{x} = \langle \text{Row}_1(I_n)\cdot\vec{x}, \text{Row}_2(I_n)\cdot\vec{x}, \ldots, \text{Row}_n(I_n)\cdot\vec{x} \rangle$$

$$= \langle \vec{e}_1\cdot\vec{x}, \vec{e}_2\cdot\vec{x}, \ldots, \vec{e}_n\cdot\vec{x} \rangle$$

$$= \langle x_1, x_2, \ldots, x_n \rangle$$

$$= \vec{x}$$

Matrix-Vector Product Multiplicative Identity

If $\vec{x}$ is any vector in R^n, then

$$I_n\vec{x} = \vec{x}.$$

The $n \times n$ identity matrix is the multiplicative identity for the matrix-vector product.

Matrix Multiplicative Identity

Recall: $A\vec{e}_i = \text{Col}_i(A)$, $\quad \text{Col}_i(AB) = A\,\text{Col}_i(B)$ $\quad$ and $\quad \text{Row}_i(AB) = B^T\,\text{Row}_i(A)$.

$\underset{m\times n}{A}\ \underset{n\times n}{I} \qquad\qquad \underset{m\times m}{I}\ \underset{m\times n}{A}$

Let A be an $m \times n$ matrix. Evaluate

$\text{Col}_i(AI_n) = A\,\text{Col}_i(I_n) = A\vec{e}_i = \text{Col}_i(A)$

$\Rightarrow AI_n = A$

and

$\text{Row}_i(I_mA) = A^T\,\text{Row}_i(I_m) = A^T\vec{e}_i = \text{Col}_i(A^T) = \text{Row}_i(A)$

$I_mA = A$

Matrix Product Multiplicative Identity

Let A be any $m \times n$ matrix. Then

$$AI_n = A, \quad \text{and} \quad I_m A = A.$$

The identity matrix (of the appropriate size) is the multiplicative identity for the product of two matrices.

The **identity** matrix is the multiplicative identity for both the matrix-vector and the matrix-matrix products. I_n is a matrix multiplication analog of the number 1.

3.7 The Associative Property of Matrix Multiplication

We know that matrix multiplication is not commutative. However, it is associative.

Associativity of Matrix Multiplication

If A is an $m \times p$ matrix, B is a $p \times q$ matrix, and C is a $q \times n$ matrix, then

$$(AB)C = A(BC).$$

Intermediate products will involve different sized matrices.

(AB) C	A (BC)
m x q q x n	m x p p x n
m x n	m x n

Associativity and the Matrix-Vector Product

Special Case

Suppose A is an $m \times p$ matrix and B is a $p \times n$ matrix. Let $\vec{x}$ be a vector in R^n. Then

$$(AB)\vec{x} = A\left(B\vec{x}\right).$$

Remark: This is a primary motivation for how the product AB of matrices is defined. This will become critical when we study **linear transformations** between R^m and R^n.

3.8 Matrix Equations

We'll consider two types of matrix equations.

Matrix-Vector Equation

$$A\vec{x} = \vec{y}$$

The matrix A and the vector $\vec{y}$ are known. The variable to be solved for is the vector $\vec{x}$.

Matrix-Matrix Equation

$$AX = Y$$

The matrices A and Y are known. The variable to be solved for is the matrix X.

Matrix-Vector Equation

Suppose $A = [a_{ij}]$ is an $m \times n$ matrix, and let $\vec{y} = \langle y_1, y_2, \ldots, y_m \rangle$ be a vector in R^m.

$$A\vec{x} = \vec{y} \tag{1}$$

- Any vector $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in R^n that satisfies equation (1) is called a **solution**.
- The **solution set** of equation (1) is the set of all vectors $\vec{x}$ in R^n that are solutions.
- Equation (1) may have no solution, exactly one solution, or infinitely many solutions.

Linear System

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $\vec{y} = \langle y_1, y_2 \rangle$. Identify the entry equations of $A\vec{x} = \vec{y}$ where $\vec{x} = \langle x_1, x_2, x_3 \rangle$.

$$A\vec{x} = \langle \text{Row}_1(A)\cdot\vec{x}, \text{Row}_2(A)\cdot\vec{x} \rangle = \langle y_1, y_2 \rangle$$

$$= \langle a_{11}x_1 + a_{12}x_2 + a_{13}x_3, a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \rangle$$

$$= \langle y_1, y_2 \rangle$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = y_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = y_2$$

$A\vec{x} = \vec{y}$ is the system of

linear equations whose

coefficient matrix is A.

$A\vec{x} = \vec{y}$

The vector $\vec{x}$ is a solution of the matrix-vector equation $A\vec{x} = \vec{y}$ if and only if $(x_1, x_2, \ldots, x_n)$ is a solution of the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= y_m. \end{aligned}$$

Note this is the system having coefficient matrix A and augmented matrix $\widehat{A} = [A \,|\, \vec{y}]$.

Theorem

Suppose that A is an $m \times n$ matrix and that $\vec{y}$ is a vector in R^m and consider the matrix–vector equation

$$A\vec{x} = \vec{y}.$$

Let $\widehat{A}$ be the augmented matrix $\widehat{A} = [\ A \mid \vec{y}\ ]$.

1. If the rightmost column of $\widehat{A}$ is a pivot column of $\widehat{A}$, then $A\vec{x} = \vec{y}$ is inconsistent.
2. If the rightmost column of $\widehat{A}$ is not a pivot column of $\widehat{A}$, then $A\vec{x} = \vec{y}$ is consistent.

Moreover, if $A\vec{x} = \vec{y}$ is consistent then

1. If every column of A is a pivot column of A, then $A\vec{x} = \vec{y}$ has a unique solution.
2. If at least one column of A is not a pivot column of A, then $A\vec{x} = \vec{y}$ has infinitely many solutions.