September 27 Math 2306 sec. 51 Fall 2021

Section 9: Method of Undetermined Coefficients

We were considering linear, constant coefficient, nonhomogeneous ODEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

At first, we are looking at the y_p part. The general solution will be $y = y_c + y_p$.

Method of Undetermined Coefficients

This is a method for finding a particular solution to

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

by assuming that y_p is the same kind of function as g. To find the general solution

- Find y_c.
- Determine what type of function g is and set up a guess for y_p of this form.
- Compare the guess to y_c and multiply by x^n if needed to get rid of common like terms.
- Substitute our guess into the ODE,
- and then solve a system of equations for the coefficients by matching like terms.

The Glitch

y_p can't share like terms in common with y_c.

$$y'' - y' = 3e^x$$

Based on $g(x) = 3e^x$

we tried $y_p = Ae^x$,

but it didnt work.

The Glitch

y_p can't share like terms in common with y_c.

$$y'' - y' = 3e^x$$

- What is y_c?

y_c solves

$$y'' - y' = 0$$

$$y_c = C_1 + C_2 e^x$$

Characteristic eqn

$$m^2 - m + 0 = 0$$

$$m^2 - m = 0$$

$$m(m-1) = 0$$

$m = 0$ or $m = 1$

$$y_1 = e^{0x} = 1$$

$$y_2 = e^{1x} = e^x$$

The Glitch

y_p can't share like terms in common with y_c.

$$y'' - y' = 3e^x$$

- What is y_c? $y_c = c_1 + c_2 e^x$

- What is the correct form for y_p? $y_p = Axe^x$

The Glitch

y_p can't share like terms in common with y_c.

$$y'' - y' = 3e^x$$

- What is y_c? $y_c = c_1 + c_2 e^x$
- What is the correct form for y_p? $y_p = Axe^x$
- Find y_p. $\boxed{y_p = 3xe^x}$

$y_p' = Ae^x + Axe^x$

$y_p'' = Ae^x + Ae^x + Axe^x$

$y_p'' - y_p' = Ae^x = 3e^x$

$A = 3$

The Glitch

y_p can't share like terms in common with y_c.

$$y'' - y' = 3e^x$$

- What is y_c? $y_c = c_1 + c_2 e^x$
- What is the correct form for y_p? $y_p = Axe^x$
- Find y_p. $y_p = 3xe^x$
- What is the general solution? $\boxed{y = c_1 + c_2 e^x + 3xe^x}$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \Rightarrow m = 2$ double root

$y_1 = e^{2x}$, $y_2 = xe^{2x}$ $\quad y_c = c_1 e^{2x} + c_2 x e^{2x}$

Let's use superposition.

Let y_{p_1} solve $y'' - 4y' + 4y = \sin(4x)$

and y_{p_2} solve $y'' - 4y' + 4y = xe^{2x}$

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

For $g_1(x) = \sin(4x)$

$$y_{p_1} = A\sin(4x) + B\cos(4x)$$ this is correct

For $g_2(x) = x e^{2x}$

$$y_{p_2} = (Cx+D)e^{2x} = Cxe^{2x} + De^{2x}$$ ✗

matches y_c

$$y_p = (Cx+D)e^{2x} \cdot x^2$$
$$= Cx^3e^{2x} + Dx^2e^{2x}$$ this is correct

So $y_p = A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Find y_c: $m^3 - m^2 + m - 1 = 0$

factor by grouping

$$m^2(m-1) + (m-1) = 0$$

$$(m^2+1)(m-1) = 0$$

$m - 1 = 0 \Rightarrow m = 1$ $\quad y_1 = e^x$

$m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$ $\quad \alpha \pm i\beta$

$$\alpha = 0 \quad \beta = 1$$

$y_2 = e^{0x}\cos x = \cos x$, $y_2 = e^{0x}\sin x = \sin x$

For y_c $y_1 = e^x$, $y_2 = \cos x$, $y_3 = \sin x$

Let's find $y_p = y_{p_1} + y_{p_2}$ where

y_{p_1} solves $y''' - y'' + y' - y = \cos x$

and y_{p_2} solves $y''' - y'' + y' - y = x^4$

$g_1(x) = \cos x$, $y_{p_1} = (A\cos x + B\sin x)x$

$= Ax\cos x + Bx\sin x$ ✓ correct

$g_2(x) = x^4$ $y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$

$$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$$

Solve the IVP

$$y'' - y = e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Find y_c: The characteristic eqn is $m^2 - 1 = 0 \Rightarrow m^2 = 1$

$m = \pm 1$ two real roots.

$y_1 = e^{x}$, $y_2 = e^{-x}$ $\quad y_c = c_1 e^{x} + c_2 e^{-x}$

Find y_p: $g(x) = e^{-x}$

$y_p = A e^{-x} \cdot x = A x e^{-x}$ ✓

Sub y_p into the ODE

$$y_p = Axe^{-x}$$

$$y_p' = Ae^{-x} - Axe^{-x}$$

$$y_p'' = -2Ae^{-x} + Axe^{-x}$$

$$y_p'' - y_p = e^{-x}$$

$$-2Ae^{-x} + \cancel{Axe^{-x}} - \cancel{Axe^{-x}} = e^{-x}$$

$$-2Ae^{-x} = e^{-x}$$

$$-2A = 1 \Rightarrow A = \frac{-1}{2}$$

$$y_p = \frac{-1}{2}xe^{-x}$$

General Solution $\quad y = y_c + y_p = c_1e^{x} + c_2e^{-x} - \frac{1}{2}xe^{-x}$

Apply $y(0) = -1$ and $y'(0) = 1$

$$y(x) = c_1 e^{x} + c_2 e^{-x} - \frac{1}{2} x e^{-x}$$

$$y'(x) = c_1 e^{x} - c_2 e^{-x} - \frac{1}{2} e^{x} + \frac{1}{2} x e^{-x}$$

$$y(0) = c_1 e^{0} + c_2 e^{0} - \frac{1}{2} \cdot 0 \, e^{0} = -1$$

$$c_1 + c_2 = -1$$

$$y'(0) = c_1 e^{0} - c_2 e^{0} - \frac{1}{2} e^{0} + \frac{1}{2} \cdot 0 \cdot e^{0} = 1$$

$$c_1 - c_2 - \frac{1}{2} = 1$$

$$c_1 - c_2 = \frac{3}{2}$$

$$c_1 + c_2 = -1$$

$$c_1 - c_2 = \frac{3}{2} \qquad \text{add}$$

$$2c_1 = \frac{1}{2} \implies c_1 = \frac{1}{4}$$

$$c_2 = -1 - c_1 = -1 - \frac{1}{4} = \frac{-5}{4}$$

The solution to the IVP

$$\boxed{y = \frac{1}{4}e^{x} - \frac{5}{4}e^{-x} - \frac{1}{2}xe^{-x}}$$