September 27 Math 2306 sec. 51 Spring 2023 Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order¹, linear, homogeneous equation with constant coefficients

$$arac{d^2y}{dx^2}+brac{dy}{dx}+cy=0, \quad ext{with } a
eq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\frac{dy}{dx} - \frac{c}{a}y.$$

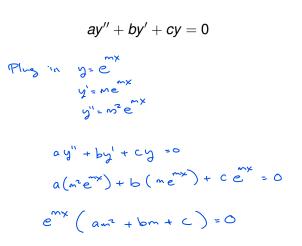
Question: What sorts of functions y could be expected to satisfy

$$y'' = (\text{constant}) y' + (\text{constant}) y?$$



¹We'll extend the result to higher order at the end of this sectionSeptember 25, 2023 1/25

We look for solutions of the form $y = e^{mx}$ with *m* constant.



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This will be true if m satisfier

 $am^2 + bm + C = 0$

Suppose *a*, *b*, and *c* are real numbers and $a \neq 0$. The function $y = e^{mx}$ solves the second order, homogeneous ODE

$$ay''+by'+cy=0$$

on $(-\infty,\infty)$ provided *m* is a solution of the quadratic equation

$$am^2+bm+c=0.$$

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Characteristic (a.k.a. Auxiliary) Equation

The characteristic equation for the second order, linear, homogeneous ODE ay'' + by' + cy = 0 is the quadratic equation

 $am^2 + bm + c = 0$

There are three cases that we must consider.

- $b^2 4ac > 0$ then there are two distinct real roots $m_1 \neq m_2$
- II $b^2 4ac = 0$ then there is one repeated real root $m_1 = m_2 = m$
- III $b^2 4ac < 0$ then there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$ where α and β are real numbers and $\beta > 0$.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac > 0$.

There are two different roots m_1 and m_2 . A fundamental solution set consists of

$$y_1 = e^{m_1 x}$$
 and $y_2 = e^{m_2 x}$.

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

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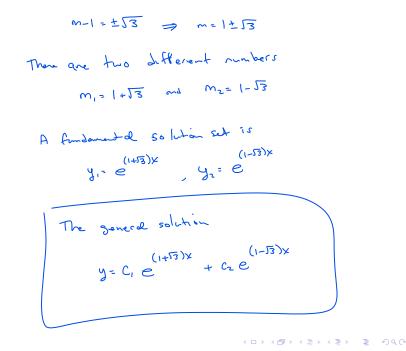
Example

Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$
The ODE is 2^{nd} order, linear, homogeneous, w) constant
coefficients. The characteristic equation is
$$m^{2} - 2m - 2 = 0$$
Find the roots.
$$m^{2} - 2m + 1 - 1 - 2 = 0$$

$$(-\frac{2}{3})^{2} - 1 \qquad (m^{2} - 2m + 1) - 3 = 0$$

$$(m - 1)^{2} = 3$$



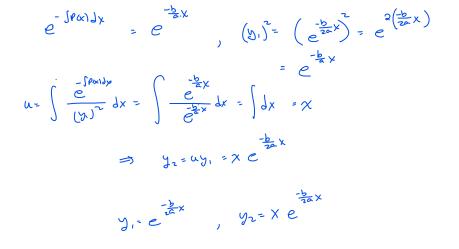
Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac = 0$

There is only one real, double root, $m = \frac{-b}{2a}$.

Use reduction of order to find the second solution to the equation (in standard form)

 $y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$ $y_2 = wy, \quad w = \int \frac{e^{-\frac{b}{2a}x}}{y_1^2} \, dx$ $P(x) = \frac{b}{a}, \quad -\int P(x) dx = -\int \frac{b}{a} \, dx = -\frac{b}{a} \, x$



Case II: One repeated real root

$$ay''+by'+cy=0,$$
 where $b^2-4ac=0$

If the characteristic equation has one real repeated root *m*, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx}$$
 and $y_2 = xe^{mx}$.

The general solution is

$$y=c_1e^{mx}+c_2xe^{mx}.$$

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Example

Solve the IVP

y'' + 6y' + 9y = 0, y(0) = 4, y'(0) = 0The ODE is 2ⁿ¹ order, hon-sureous, constant coef and lineor. Characteristic egn is $m^{2} + 6m + 9 = 0 \implies (m + 3)^{2} = 0$ ⇒ m=-3 repeated root. y, = e , y2 = x e 3x The general solution is y= C, e^{3x} + Cz × e^{3x} September 25, 2023

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Apply
$$y_{(6)=4}$$
, $y'_{(6)=0}$
 $y'=-3c, e^{3x} + 6e^{3x} - 3c_{2}xe^{3x}$
 $y_{(6)}=c, e^{6} + c_{2} \circ e^{6} = 4 \implies c_{1}=4$
 $y'_{(6)}=-3c, e^{6} + c_{2} e^{6} - 3c_{2} \circ \cdot e^{6} = 0$
 $-3c_{1} + c_{2} = 0 \implies c_{2}=3c_{1} = 12$
The solution to the IVP is
 $y_{2}=4e^{-3x} + 12xe^{-3x}$

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