September 27 Math 2306 sec. 51 Spring 2023

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0, \quad \text{with } a\neq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2}=-\frac{b}{a}\frac{dy}{dx}-\frac{c}{a}y.$$

Question: What sorts of functions y could be expected to satisfy

$$y''=(\text{constant})\,y'+(\text{constant})\,y?$$

Exponential e^{mx} , sines/cosines $\cos(kx)$ or $\sin(kx)$

m - constant

k - constant

polynomials

[1]We'll extend the result to higher order at the end of this section.

We look for solutions of the form $y = e^{mx}$ with m constant.

$$ay'' + by' + cy = 0$$

Plug in $y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

$ay'' + by' + cy = 0$

$a(m^2 e^{mx}) + b(me^{mx}) + ce^{mx} = 0$

$e^{mx}(am^2 + bm + c) = 0$

This will be true if m satisfies

$$am^2 + bm + c = 0$$

Suppose a, b, and c are real numbers and $a \neq 0$. The function $y = e^{mx}$ solves the second order, homogeneous ODE

$$ay'' + by' + cy = 0$$

on $(-\infty, \infty)$ provided m is a solution of the quadratic equation

$$am^2 + bm + c = 0.$$

Characteristic (a.k.a. Auxiliary) Equation

The characteristic equation for the second order, linear, homogeneous ODE $ay'' + by' + cy = 0$ is the quadratic equation

$$am^2 + bm + c = 0$$

There are three cases that we must consider.

I $b^2 - 4ac > 0$ then there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ then there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ then there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$ where α and β are real numbers and $\beta > 0$.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Example

Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$

The ODE is 2^{nd} order, linear, homogeneous, w/ constant coefficients. The characteristic equation is

$$m^2 - 2m - 2 = 0$$

$$am^2 + bm + c = 0$$

Find the roots.

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m^2 - 2m + 1 - 1 - 2 = 0$$

$$a\left(m^2 + \frac{b}{a}m + \frac{c}{a}\right) = 0$$

$$\left(\frac{-2}{2}\right)^2 = 1$$

$$(m^2 - 2m + 1) - 3 = 0$$

$$a\left(m + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a^2} = 0$$

$$(m-1)^2 = 3$$

$$m-1 = \pm\sqrt{3} \implies m = 1 \pm \sqrt{3}$$

There are two different numbers

$$m_1 = 1+\sqrt{3} \quad \text{and} \quad m_2 = 1-\sqrt{3}$$

A fundamental solution set is

$$y_1 = e^{(1+\sqrt{3})x}, \quad y_2 = e^{(1-\sqrt{3})x}$$

The general solution

$$y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

There is only one real, double root, $m = \dfrac{-b}{2a}$.

Use reduction of order to find the second solution to the equation (in standard form)

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$$

$y_2 = uy_1$ where $u = \displaystyle\int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$

$P(x) = \dfrac{b}{a}$, $-\displaystyle\int P(x)\,dx = -\int \frac{b}{a}\,dx = -\frac{b}{a}x$

$$e^{-\int p(x)\,dx} = e^{-\frac{b}{a}\cdot x}, \quad (y_1)^2 = \left(e^{\frac{-b}{2a}x}\right)^2 = e^{2\left(\frac{-b}{2a}x\right)}$$

$$= e^{-\frac{b}{a}x}$$

$$u = \int \frac{e^{-\int p(x)\,dx}}{(y_1)^2}\,dx = \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx = \int dx = x$$

$$\Rightarrow \quad y_2 = u y_1 = x\, e^{-\frac{b}{2a}x}$$

$$y_1 = e^{-\frac{b}{2a}x}, \quad y_2 = x\, e^{-\frac{b}{2a}x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 xe^{mx}.$$

Example

Solve the IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

The ODE is 2^{nd} order, homogeneous, constant coef. and linear.

Characteristic eqn is

$$m^2 + 6m + 9 = 0 \implies (m+3)^2 = 0$$

$\Rightarrow$ $m = -3$ repeated root.

$$y_1 = e^{-3x}, \quad y_2 = x e^{-3x}$$

The general solution is $y = c_1 e^{-3x} + c_2 x e^{-3x}$

Apply $y(0)=4$, $y'(0)=0$

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 e^0 = 4 \Rightarrow c_1 = 4$$

$$y'(0) = -3c_1 e^0 + c_2 e^0 - 3c_2 \cdot 0 \cdot e^0 = 0$$

$$-3c_1 + c_2 = 0 \Rightarrow c_2 = 3c_1 = 12$$

The solution to the IVP is

$$y = 4e^{-3x} + 12x e^{-3x}$$