September 27 Math 2306 sec. 52 Fall 2021

Section 9: Method of Undetermined Coefficients

We were considering linear, constant coefficient, nonhomogeneous ODEs
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$
where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

At first, we are looking at the y_p part. The general solution will be $y = y_c + y_p$.

Method of Undetermined Coefficients

This is a method for finding a particular solution to

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

by assuming that y_p is the same kind of function as g. To find the general solution

- Find y_c.
- Determine what type of function g is and set up a guess for y_p of this form.
- Compare the guess to y_c and multiply by x^n if needed to get rid of common like terms.
- Substitute our guess into the ODE,
- and then solve a system of equations for the coefficients by matching like terms.

The Glitch

y_p can't share like terms in common with y_c.

$$y'' - y' = 3e^x$$

Here $g(x) = 3e^x$

we guessed $y_p = Ae^x$

But it didn't work.

The Glitch

y_p can't share like terms in common with y_c.

$$y'' - y' = 3e^x$$

- What is y_c? y_c solves $y'' - y' = 0$

Characteristic eqn $m^2 - m = 0$

$m(m-1) = 0$

$m = 0$ or $m - 1 = 0$ i.e. $m = 1$

$y_1 = e^{0x} = 1$, $y_2 = e^x$ $y_c = c_1 + c_2 e^x$

The Glitch

y_p can't share like terms in common with y_c.

$$y'' - y' = 3e^x$$

- What is y_c? $y_c = c_1 + c_2 e^x$
- What is the correct form for y_p? $y_p = Axe^x$ ✓ correct

The Glitch

y_p can't share like terms in common with y_c.

$$y'' - y' = 3e^x$$

- What is y_c? $y_c = c_1 + c_2 e^x$
- What is the correct form for y_p? $y_p = Axe^x$
- Find y_p.

$y_p = 3xe^x$

$y_p' = Ae^x + Axe^x$

$y_p'' = Ae^x + Ae^x + Axe^x$

$y_p'' - y_p' = Ae^x = 3e^x \quad A = 3$

The Glitch

y_p can't share like terms in common with y_c.

$$y'' - y' = 3e^x$$

- What is y_c? $y_c = c_1 + c_2 e^x$
- What is the correct form for y_p? $y_p = Axe^x$

$y = y_c + y_p$

- Find y_p. $y_p = 3xe^x$
- What is the general solution? $y = c_1 + c_2 e^x + 3xe^x$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: Characteristic eqn

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0 \Rightarrow m = 2 \quad \text{double root}$$

$$y_1 = e^{2x}, \quad y_2 = xe^{2x} \quad \text{so} \quad y_c = c_1 e^{2x} + c_2 x e^{2x}$$

Using superposition, set $y_p = y_{p_1} + y_{p_2}$

where y_{p_1} solves $y'' - 4y' + 4y = \sin(4x)$

and y_{p_2} solves $y'' - 4y' + 4y = xe^{2x}$

For $g_1(x) = \sin(4x)$

$$y_{p_1} = A\sin(4x) + B\cos(4x)$$

correct form

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

For $g_2(x) = xe^{2x}$

$$y_{p_2} = (Cx + D)e^{2x} = Cxe^{2x} + De^{2x}$$

matches y_c

$$y_{p_2} = (Cx + D)e^{2x}x^2$$

$$= (Cx^3 + Dx^2)e^{2x} = Cx^3e^{2x} + Dx^2e^{2x}$$

$$y_p = A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}$$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Find y_c: Characteristic eqn

$$m^3 - m^2 + m - 1 = 0$$

$$m^2(m-1) + (m-1) = 0$$

$$(m-1)(m^2+1) = 0$$

$m - 1 = 0 \Rightarrow m = 1 \qquad y_1 = e^{1x} = e^x$

$m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i \qquad \alpha \pm i\beta$

$\alpha = 0 \quad \beta = 1$

$y_2 = e^{0x}\cos x = \cos x, \quad y_3 = e^{0x}\sin x = \sin x$

$y_1 = e^x$, $y_2 = \cos x$, $y_3 = \sin x$

$$y_c = C_1 e^x + C_2 \cos x + C_3 \sin x$$

Let $y_p = y_{p_1} + y_{p_2}$ where

y_{p_1} solves $y''' - y'' + y' - y = \cos x$

$g_1(x) = \cos x$, $y_{p_1} = (A \cos x + B \sin x)x$

$= Ax \cos x + Bx \sin x$

y_{p_2} solve $y''' - y'' + y' - y = x^4$

$g_2(x) = x^4$, $y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$

$$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G.$$

Solve the IVP

$$y'' - y = e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Find y_c: Characteristic eqn

$$m^2 - 1 = 0 \Rightarrow m^2 = 1$$

$$m = \pm 1$$

$$y_1 = e^{-x}, \quad y_2 = e^{x} \qquad y_c = C_1 e^{-x} + C_2 e^{x}$$

Find y_p: $g(x) = e^{-x}$

$$y_p = (A e^{-x})x = A x e^{-x} \quad \checkmark$$

Sub to find A.

$$y_p = Axe^{-x}$$

$$y_p' = Ae^{-x} - Axe^{-x}$$

$$y_p'' = -2Ae^{-x} + Ax.e^{-x}$$

$$y_p'' - y_p = e^{-x}$$

$$-2Ae^{-x} + A\cancel{xe^{-x}} - A\cancel{xe^{-x}} = e^{-x}$$

$$-2Ae^{-x} = e^{-x}$$

$$-2A = 1$$

$$A = \frac{-1}{2}$$

$$y_p = \frac{-1}{2}xe^{-x}$$

The general solution

$$y = c_1 e^{-x} + c_2 e^{x} - \frac{1}{2} x e^{-x}$$

Now apply $y(0) = -1$, $y'(0) = 1$

$$y' = -c_1 e^{-x} + c_2 e^{x} - \frac{1}{2} e^{-x} + \frac{1}{2} x e^{-x}$$

$$y(0) = c_1 e^{0} + c_2 e^{0} - \frac{1}{2} \cdot 0 \cdot e^{0} = -1$$

$$c_1 + c_2 = -1$$

$$y'(0) = -c_1 e^{0} + c_2 e^{0} - \frac{1}{2} e^{0} + \frac{1}{2} \cdot 0 \cdot e^{0} = 1$$

$$-c_1 + c_2 - \frac{1}{2} = 1$$

$$-c_1 + c_2 = \frac{3}{2}$$

Solve

$$c_1 + c_2 = -1$$
$$-c_1 + c_2 = \frac{3}{2}$$

add

$$2c_2 = \frac{1}{2} \Rightarrow c_2 = \frac{1}{4}$$

subtract

$$2c_1 = -\frac{5}{2} \Rightarrow c_1 = \frac{-5}{4}$$

The solution to the IVP

$$\boxed{y = \frac{-5}{4}e^{-x} + \frac{1}{4}e^{x} - \frac{1}{2}xe^{-x}}$$