September 28 Math 2306 sec. 51 Fall 2022

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example[1]

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

Constant coefficient left, polynomial right.

$g(x) = 8x+1$ 1st degree polynomial

Let's guess that y_p is also a 1st degree polynomial.

Set $y_p = Ax + B$ w/ A, B constants

Sub into the ODE. $y_p'' - 4y_p' + 4y_p = 8x+1$

[1] **We're only ignoring the y_c part to illustrate the process.**

$y_p = Ax+B$

$y_p' = A$

$y_p'' = 0$

$$0 - 4(A) + 4(Ax+B) = 8x+1$$

$$4Ax + (-4A+4B) = 8x+1$$

Matching coefficients

$$4A = 8 \implies A = 2$$

$$-4A + 4B = 1 \implies 4B = 1+4A$$

$$B = \frac{1}{4}(1+4A)$$

$$= \frac{1}{4}(1+8) = \frac{9}{4}$$

$y_p = 2x + \frac{9}{4}$ is a particular solution

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We found the particular solution

$$y_p = 2x + \frac{9}{4}$$

by

- guessing that y_p is the same kind of function as g,
- setting it up with **undetermined** coefficients (A, B, etc.), and
- substituting it into the ODE to find the coefficients that work.

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

$g(x) = 6e^{-3x}$ is an exponential

Set $y_p = Ae^{-3x}$. Sub into the ODE.

$y_p' = -3Ae^{-3x}$

$y_p'' = 9Ae^{-3x}$

$$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$$

$$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4Ae^{-3x} = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x}$$

Matching gives $25A = 6$

$$A = \frac{6}{25}$$

We found $y_p = \frac{6}{25}e^{-3x}$

The Initial Guess Must Be General in Form

Find a particular solution to $y'' - 4y' + 4y = 16x^2$

Here $g(x) = 16x^2$. This can be thought of as ① a monomial in x^2

or ② a 2^{nd} degree polynomial

Thinking of g as a constant times x^2,

Set $y_p = Ax^2$ Substitute

$y_p' = 2Ax$

$y_p'' = 2A$ $y_p'' - 4y_p' + 4y_p = 16x^2$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

Matching gives

$$\left.\begin{aligned} 4A &= 16 \\ -8A &= 0 \\ 2A &= 0 \end{aligned}\right\}$$ not possible as $A = 4$ and $A = 0$ can't both be true.

We didn't account for the like terms, x and constant that arise from differentiation. We should thing of $g(x) = 16x^2$ as a 2^{nd} degree polynomial.

Try $\quad y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

Match

$$4A = 16$$

$$-8A + 4B = 0$$

$$2A - 4B + 4C = 0$$

Solve $\quad A = 4$

$$4B = 8A \quad \Rightarrow \quad B = 2A = 8$$

$$4C = -2A + 4B = -2(4) + 4(8) = 24$$

$$\Rightarrow \quad C = 6.$$

We find $\boxed{y_p = 4x^2 + 8x + 6}$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Examples of Forms of y_p based on g (Trial Guesses)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

(a) $g(x) = 1$ (or really any nonzero constant)

$$y_p = A$$

(b) $g(x) = x - 7$ (1^{st} degree polynomial)

$$y_p = Ax + B$$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ (2[nd] degree polynomial)

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ (3[rd] degree polynomial)

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$ (1st degree polynomial times e^{3x})

$$y_p = (Ax+B)e^{3x}$$

$$= Axe^{3x} + Be^{3x}$$

(f) $g(x) = \cos(7x)$ (linear combo of cosine and sine of $7x$)

$$y_p = A\cos(7x) + B\sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$ (two linear combos of sine/cosine)

$$y_p = A\sin(2x) + B\cos(2x) + C\cos(4x) + D\sin(4x)$$

(h) $g(x) = x^2\sin(3x)$ (linear combo 2^{nd} degree polynomial time sine and 2^{nd} degree poly times cosine)

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$ (linear combo of e^x cosine and e^x sine of $2x$)

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$ (linear combo of 1^{st} poly times e^{-x} sine and 1^{st} poly times e^{-x} cosine)

$$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$$