September 28 Math 2306 sec. 52 Fall 2022

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example[1]

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

The left side is constant coefficient, and the right hand side $g(x) = 8x + 1$ is a **first degree polynomial**. We made a guess that the particular solution y_p was also a first degree polynomial.

$$y_p = Ax + B \quad \text{for } A \text{ and } B \text{ constants.}$$

Subsitution led to finding the numbers

$$A = 2 \quad \text{and} \quad B = \frac{9}{4}.$$

[1] **We're only ignoring the y_c part to illustrate the process.**

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We found the particular solution

$$y_p = 2x + \frac{9}{4}$$

by

- guessing that y_p is the same kind of function as g,
- setting it up with **undetermined** coefficients (A, B, etc.), and
- substituting it into the ODE to find the coefficients that work.

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

$g(x) = 6e^{-3x}$ a constant times e^{-3x}

Let $y_p = Ae^{-3x}$. Substitute

$y_p' = -3Ae^{-3x}$

$y_p'' = 9Ae^{-3x}$

$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$

$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4Ae^{-3x} = 6e^{-3x}$

$$25A\, e^{-3x} = 6\, e^{-3x}$$

Matching gives $\quad 25A = 6$

$$\Rightarrow \quad A = \frac{6}{25}$$

$$\boxed{\text{We find } \quad y_p = \frac{6}{25} e^{-3x}}$$

y_c solves the associated homogeneous ODE

$$y'' - 4y' + 4y = 0$$

y_p solves the nonhomogeneous eqn.

The Initial Guess Must Be General in Form

Find a particular solution to $y'' - 4y' + 4y = 16x^2$

Here $g(x) = 16x^2$. This can be classified in a couple of ways.

① it's a monomial, constant times x^2

② it's a 2nd degree polynomial

Considering a monomial, suppose we set

$y_p = Ax^2$

$y_p' = 2Ax$

$y_p'' = 2A$

$y_p'' - 4y_p' + 4y_p = 16x^2$

$$2A - 4(2Ax) + 4Ax^2 = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

Matching like terms

$$\left.\begin{aligned} 4A &= 16 \\ -8A &= 0 \\ 2A &= 0 \end{aligned}\right\}$$

can't be solved as it requires $A=4$ and $A=0$

We didn't account for x or constant terms.

Consider $g(x) = 16x^2$ as a 2nd degree polynomial.

Set $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

Match like terms

$$4A = 16 \Rightarrow A = 4$$

$$-8A + 4B = 0$$

$$2A - 4B + 4C = 0$$

$$4B = 8A \Rightarrow B = 2A = 2(4) = 8$$

$$4C = -2A + 4B = -2(4) + 4(8) = 24 \Rightarrow C = 6$$

$$\boxed{\text{we set } y_p = 4x^2 + 8x + 6}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Examples of Forms of y_p based on g (Trial Guesses)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

(a) $g(x) = 1$ (or really any nonzero constant)

$y_p = A$

(b) $g(x) = x - 7$ (1st degree polynomial)

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ (2nd degree polynomial)

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ (3rd degree polynomial)

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$ (1st degree polynomial times e^{3x})

$$y_p = (Ax+B)e^{3x}$$
$$= Axe^{3x} + Be^{3x}$$

(f) $g(x) = \cos(7x)$ (linear combo of cosine and sine of $7x$)

$$y_p = A\cos(7x) + B\sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$ (two linear combos of sine/cosine)

$$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$$

(h) $g(x) = x^2 \sin(3x)$ (linear combo 2nd degree polynomial time sine and 2nd degree poly times cosine)

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$ (linear combo of e^x cosine and e^x sine of $2x$)

$$y_p = A e^x \cos(2x) + B e^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$ (linear combo of 1st poly times e^{-x} sine and 1st poly times e^{-x} cosine)

$$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$$

The Superposition Principle

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

The principle of superposition for nonhomogeneous equations tells us that we can find y_p by considering separate problems

$$y_{p_1} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x)$$

$$y_{p_2} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_2(x),$$

and so forth.

Then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$.

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

Find y_{p_1} that solves

$$y'' - 4y' + 4y = 6e^{-3x}$$

we know $y_{p_1} = Ae^{-3x}$

Find y_{p_2} that solves

$$y'' - 4y' + 4y = 16x^2$$

$$y_{p_2} = Bx^2 + Cx + D$$

The form for y_p is

$$y_p = Ae^{-3x} + Bx^2 + Cx + D$$