September 29 Math 2306 sec. 51 Spring 2023

Section 8: Homogeneous Equations with Constant Coefficients

We are considering the second order, linear, homogeneous ODE with constant coefficients

$$ay'' + by' + cy = 0.$$

The function $y = e^{mx}$ is a solution provided m is a solution of the **characteristic equation**

$$am^2 + bm + c = 0.$$

We have to consider three cases,

- **Case I:** there are two different real roots, $m_1 \neq m_2$,
- **Case II:** there is one repeated real root, $m_1 = m_2 = m$,
- **Case III:** the roots are complex conjugates $m = \alpha \pm i\beta$ with $\beta > 0$.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx}.$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

The two roots of the characteristic equation will be

$$m_1 = \alpha + i\beta \quad \text{and} \quad m_2 = \alpha - i\beta \quad \text{where} \quad i^2 = -1.$$

We want our solutions in the form of *real valued* functions. We start by writing a pair of solutions

$$Y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x}.$$

We will use the **principle of superposition** to write solutions y_1 and y_2 that do not contain the complex number i.

Deriving the solutions Case III

Recall Euler's Formula[1] : $e^{i\theta} = \cos\theta + i\sin\theta$.

$\theta = \beta x$

$$Y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$
$$= e^{\alpha x}\cos(\beta x) + i e^{\alpha x}\sin(\beta x)$$

$$Y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$
$$= e^{\alpha x}\cos(\beta x) - i e^{\alpha x}\sin(\beta x)$$

Set $y_1 = \frac{1}{2}(Y_1 + Y_2) = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$

$y_2 = \frac{1}{2i}(Y_1 - Y_2) = \frac{1}{2i}\left(2i\, e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$

[1] As the sine is an odd function $e^{-i\theta} = \cos\theta - i\sin\theta$.

In the complex case, we must identify the α and β.

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

Let α be the real part of the complex roots and $\beta > 0$ be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Example

Find the general solution of $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 6x = 0.$

The equation is linear, homogeneous, w/ constant coef. The characteristic equation is

$$m^2 + 4m + 6 = 0.$$

Find the roots

$$m = \frac{-4 \pm \sqrt{4^2 - 4(1)(6)}}{2(1)} = \frac{-4 \pm \sqrt{16-24}}{2}$$

$$= \frac{-4 \pm \sqrt{-8}}{2} = \frac{-4 \pm 2\sqrt{2}\,i}{2}$$

$$= -2 \pm \sqrt{2}\,i$$

Complex case w/ $\alpha = -2$ and $\beta = \sqrt{2}$.

A fundamental solution set is

$$x_1 = e^{-2t}\cos(\sqrt{2}\,t), \quad x_2 = e^{-2t}\sin(\sqrt{2}\,t)$$

The general solution is

$$x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.

- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ for each pair of complex roots.

- It may require a computer algebra system to find the roots for a high degree polynomial.

Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, m may be a root of multiplicity k where $1 \leq k \leq n$.

- If a real root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \ldots, \quad x^{k-1} e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

Example

Find the general solution of the ODE.

$y''' - 3y'' + 3y' - y = 0$

3rd order, linear, homogeneous, w/ constant coef.

The characteristic equation is

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0 \implies m = 1 \quad \text{triple root}$$

A fundamental solution set is

$$y_1 = e^{1x}, \quad y_2 = x e^{1x}, \quad y_3 = x^2 e^{1x}$$

The general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

$$\begin{array}{r|l} & m^2 - 2m + 1 \\ \hline m-1 & m^3 - 3m^2 + 3m - 1 \end{array}$$

$-3m^2$ $+3m$

$m^3 - 2m^2 + m - m^2 + 2m - 1$

$= m(m^2 - 2m + 1) - (m^2 - 2m + 1)$

$= m(m-1)^2 - 1(m-1)^2 = (m-1)(m-1)^2$

Example

Find the general solution of the ODE.

$y^{(4)} + 3y'' - 4y = 0$

The char. eqn is $m^4 + 3m^2 - 4 = 0$

$(m^2+4)(m^2-1) = 0$

$(m^2+4)(m-1)(m+1) = 0$

$m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \pm\sqrt{-4} = \pm 2i = 0 \pm 2i$

$m - 1 = 0 \Rightarrow m = 1$

$m + 1 = 0 \Rightarrow m = -1$

$m_{1,2} = \pm 2i$, $y_1 = e^{0x}\cos(2x)$, $y_2 = e^{0x}\sin(2x)$

$y_3 = e^{1x}$, $y_4 = e^{-1x}$

The gen. solution

$$y = C_1 \cos(2x) + C_2 \sin(2x) + C_3 e^{x} + C_4 e^{-x}$$