September 29 Math 2306 sec. 52 Fall 2021

Question: Could the method of undetermined coefficients be used to find particular solutions to the ODEs

$$y'' + y = \tan x, \quad \textbf{or} \quad x^2y'' + xy' - 4y = e^x?$$

(Why or why not?)

Doesn't work for this type of right side

Left side is not constant coefficient.

Section 10: Variation of Parameters

We're still interested in linear, nonhomogeneous ODEs. Something of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The previous method (undetermined coefficients) can only be applied if the left side is constant coefficient and the right side is a of a certain type.

If we want to solve an equation that doesn't have a constant coefficient left

$$\text{e.g.}\quad x^2y'' + xy' - 4y = e^x,$$

or doesn't have the correct sort of right hand side

$$\text{e.g.}\quad y'' + y = \tan x,$$

we'll need another solution process.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

$y_c = c_1 y_1(x) + c_2 y_2(x)$

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have two unknowns, u_1 and u_2, but only one equation, the ODE. We'll introduce a 2nd equation.

Sub y_p into the ODE.

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

Assume that $u_1' y_1 + u_2' y_2 = 0$

This will be our 2^{nd} equation.

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

$$y_p'' + P(x) y_p' + Q(x) y_p = g(x)$$

$$u_1' y_1' + u_2' y_2' + \underline{u_1 y_1''} + \underline{u_2 y_2''} + P(x)\left(\underline{u_1 y_1'} + \underline{u_2 y_2'}\right) + Q(x)\left(\underline{u_1 y_1} + \underline{u_2 y_2}\right) = g(x)$$

Collect u_1', u_2', u_1 and u_2

$$u_1' y_1' + u_2' y_2' + \underbrace{\left(y_1'' + P(x) y_1' + Q(x) y_1\right)}_{=0} u_1 + \underbrace{\left(y_2'' + P(x) y_2' + Q(x) y_2\right)}_{=0} u_2 = g(x)$$

recall $\quad y_i'' + P(x) y_i' + Q(x) y_i = 0 \quad i = 1$ or 2

This reduces to

$$u_1' y_1' + u_2' y_2' = g(x)$$

Combine this with $\quad u_1' y_1 + u_2' y_2 = 0$

our system is
$$\boxed{\begin{array}{l} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(x) \end{array}}$$

We'll solve w/ Cramer's rule.

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

that's the Wronskian matrix

Let $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = -g y_2$.

and $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = y_1 g$.

Letting W be the Wronskian of y_1 and y_2

$$u_1' = \frac{W_1}{W} = \frac{-g y_2}{W}$$

$$u_2' = \frac{W_2}{W} = \frac{y_1 g}{W}$$

$$u_1 = \int \frac{-g(x) y_2(x)}{W} \, dx$$

$$u_2 = \int \frac{g(x) y_1(x)}{W} \, dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Example:

Solve the ODE $y'' + y = \tan x$.

$y = y_c + y_p$

Find y_c : $y_c'' + y_c = 0$

Characteristic eqn $m^2 + 1 = 0$

$m^2 = -1 \implies m = \pm i$

$\alpha = 0 \quad , \beta = 1$

$y_1 = \cos x \quad , \quad y_2 = \sin x$

$y_c = c_1 \cos x + c_2 \sin x$

To find y_p, we need g and W.

Here, $g(x) = \tan x$

$y_1 = \cos x \quad , \quad y_2 = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u_1 = \int \frac{-g y_2}{W} dx = \int \frac{-\tan x \sin x}{1} dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx = -\int \frac{1 - \cos^2 x}{\cos x} dx$$

$$= \int (\cos x - \sec x) \, dx$$

$$u_1 = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \frac{g y_1}{W}\, dx = \int \frac{\tan x \cos x}{1}\, dx$$

$$= \int \sin x\, dx$$

$$u_2 = -\cos x$$

$$y_1 = \cos x \quad , \quad y_2 = \sin x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\sin x - \ln|\sec x + \tan x|\right)\cos x - \cos x \sin x$$

$$y_p = \sin x \cancel{\cos x} - \cos x \ln|\sec x + \tan x| - \cos x \cancel{\sin x}$$

$$= -\cos x \ln|\sec x + \tan x|$$

The general solution

$$\boxed{y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|}$$

$$y'' + y = \tan x$$