# September 29 Math 2306 sec. 52 Fall 2021

## Question: Could the method of undetermined coefficients be used to find particular solutions to the ODEs

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x?$$
(Why or why not?)
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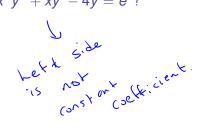
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$$y'' +$$



### Section 10: Variation of Parameters

We're still interested in linear, nonhomogeneous ODEs. Something of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The previous method (undetermined coefficients) can only be applied if the left side is constant coefficient and the right side is a of a certain type.

If we want to solve an equation that doesn't have a constant coefficient left

e.g. 
$$x^2y'' + xy' - 4y = e^x$$
,

or doesn't have the correct sort of right hand side

e.g. 
$$y'' + y = \tan x$$
,

we'll need another solution process.



#### Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose  $\{y_1(x), y_2(x)\}$  is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where  $u_1$  and  $u_2$  are functions we will determine (in terms of  $y_1$ ,  $y_2$  and g).  $\bigvee_{c} = C_1 \bigvee_{c} (x) + C_2 \bigvee_{c} (x)$ 

This method is called variation of parameters.

# Variation of Parameters: Derivation of $y_p$

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set 
$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

We have two unknowns,  $u_1$  and  $u_{2,j}$  but only

one; equation, the ODE. We'll

introduce a  $2^{nd}$  equation.

Sub yp into the ODE.

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$$

Remember that  $y_i'' + P(x)y_i' + Q(x)y_i = 0$ , for  $i = 1, 2$ 

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Assume that u,'y, + uz'yz =0

This will be our 2nd equation.

u,'y,'+ u,'y,'+ u, y','+ u, y,''+ P(x) (u,y,'+ u,y,') + Q(x) (u,y,+u,y,)= 900)

Collect u,', u, as u,

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$$u_{1}'y_{1}' + u_{2}'y_{2}' + (y_{1}'' + P(x)y_{1}' + Q(x)y_{1})u_{1} + (y_{2}'' + P(x)y_{2}' + Q(x)y_{2})u_{2} = 9 (x)$$

$$(y_{2}'' + P(x)y_{2}' + Q(x)y_{2})u_{2} = 9 (x)$$
Tecall  $y_{1}'' + P(x)y_{2}' + Q(x)y_{2} = 0$   $i=1$  or  $2$ 

This reduces to

$$u_1'y_1' + u_2'y_2' = g(x)$$

Combine this with  $u'_1y_1 + u'_2y_2 = 0$ our system is  $u'_1y_1 + u'_2y_2 = 0$   $u'_1y_1 + u'_2y_2 = 0$   $u'_1y_1 + u'_2y_2 = 0$ 

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Well solve w/ Crammers rule.

$$\begin{cases} y_1, & y_2 \\ y_1', & y_2' \end{cases} = \begin{bmatrix} 0 \\ y_2' \end{bmatrix}$$

Let 
$$W_1 = \begin{vmatrix} 0 & y_2 \\ 9 & y_1 \end{vmatrix} = -9 y_2$$

Litting W be the wronstian of y, and bz

$$\alpha_1' = \frac{M}{M'} = \frac{M}{-3\lambda^5}$$

$$u_{1} = \left( -\frac{g(x)y_{2}(x)}{\sqrt{2}} \right) \times \frac{1}{2}$$

$$u_z = \int \frac{g(x) y_1(x)}{w} dx$$

 $W = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'$ 

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# Example:

Solve the ODE  $y'' + y = \tan x$ .

Find 
$$y_c$$
:  $y_c'' + y_c = 0$   
Characteristic eqn  $m^2 + 1 = 0$   
 $m^2 = -1 \implies m = \pm i$   
 $a = 0$ ,  $\beta = 1$ 

To find yp, we need g and W.

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u_1 = \int \frac{-9y_z}{v^3} dx = \int \frac{+\infty \times \sin x}{1} dx$$

$$= - \int \frac{\sin^2 x}{\cos x} dx = - \int \frac{1 - \cos^2 x}{\cos x} dx$$

$$U_1 = \sin x - \ln |\sec x + \tan x|$$

$$U_2 = \int \frac{9y_1}{w} dx = \int \frac{\tan x \cdot Gsx}{1} dx$$

$$= \int Smx \, dx$$

$$U_2 = -Gsx$$