# September 29 Math 2306 sec. 54 Fall 2021

**Question: Could the method of undetermined coefficients be used to find particular solutions to the ODEs**

$$y'' + y = \tan x, \quad \textbf{or} \quad x^2y'' + xy' - 4y = e^x?$$

**(Why or why not?)**

the right side is not the correct type

not constant coefficient left.

## Section 10: Variation of Parameters

We're still interested in linear, nonhomogeneous ODEs. Something of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

The previous method (undetermined coefficients) can only be applied if the left side is constant coefficient and the right side is a of a certain type.

If we want to solve an equation that doesn't have a constant coefficient left

$$\text{e.g.} \quad x^2y'' + xy' - 4y = e^x,$$

or doesn't have the correct sort of right hand side

$$\text{e.g.} \quad y'' + y = \tan x,$$

we'll need another solution process.

# Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where $u_1$ and $u_2$ are functions we will determine (in terms of $y_1$, $y_2$ and $g$).

$y_c = c_1 y_1(x) + c_2 y_2(x)$

This method is called **variation of parameters**.

# Variation of Parameters: Derivation of $y_p$

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have 2 unknowns, $u_1$ and $u_2$, but only one equation, the ODE. We'll introduce a second one.

We'll sub $y_p$ into the ODE

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

Assume $u_1'y_1 + u_2'y_2 = 0$.

This is our other equation.

$$y_p = u_1y_1 + u_2y_2$$

$$y_p' = u_1y_1' + u_2y_2'$$

$$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$$

Sub into

$$y_p'' + P(x)y_p' + Q(x)y_p = g(x)$$

$$u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' + P(x)(u_1y_1' + u_2y_2') + Q(x)(u_1y_1 + u_2y_2) = g(x)$$

Collect $u_1'$, $u_2'$, $u_1$ and $u_2$

$$u_1'y_1' + u_2'y_2' + \big(\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{0}\big)u_1 +$$

$$\big(\underbrace{y_2'' + P(x)y_2' + Q(x)y_2}_{0}\big)u_2 = g(x)$$

$$y_i'' + P(x)y_i' + Q(x)y_i = 0 \qquad i = 1 \text{ or } 2$$

This reduces to

$$u_1'y_1' + u_2'y_2' = g(x)$$

Combine this with the previous equation.

Solve the system

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = g$$

We'll solve w/ Crammer's rule.

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

that's the wronskian matrix

Let $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = -g y_2$

and $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = y_1 g$.

$$u_1' = \frac{W_1}{W} = \frac{-gy_2}{W} \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

and

$$u_2' = \frac{W_2}{W} = \frac{y_1 g}{W}$$

is the Wronskian of $y_1$ and $y_2$

Hence

$$u_1 = \int \frac{-gy_2}{W}\,dx$$

and

$$u_2 = \int \frac{gy_1}{W}\,dx$$

# Example:

Solve the ODE $y'' + y = \tan x$.

$y = y_c + y_p$

Find $y_c$: $y_c'' + y_c = 0$

Characteristic eqn $m^2 + 1 = 0$

$m^2 = -1 \Rightarrow m = \pm i$

$\alpha = 0, \quad \beta = 1$

$y_1 = \cos x \qquad y_2 = \sin x$

$y_c = C_1 \cos x + C_2 \sin x$

To find $y_p$, we need $g$ and $W$.

Here, $g(x) = \tan x$.

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$y_1 = \cos x \quad , \quad y_2 = \sin x$$

$$u_1 = \int \frac{-g y_2}{W}\, dx = \int - \frac{\tan x \sin x}{1}\, dx$$

$$= - \int \frac{\sin^2 x}{\cos x}\, dx = - \int \frac{1 - \cos^2 x}{\cos x}\, dx$$

$$= \int (\cos x - \sec x)\, dx$$

$$u_1 = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \frac{g y_1}{W}\,dx = \int \frac{\tan x \cos x}{1}\,dx$$

$$= \int \sin x\,dx$$

$$= - \cos x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = (\sin x - \ln|\sec x + \tan x|)\cos x - \cos x \sin x$$

$$= \sin x \cancel{\cos x} - \cos x \ln|\sec x + \tan x| - \cos x \cancel{\sin x}$$

$$y_p = -\cos x \ln|\sec x + \tan x|$$

$$\boxed{\begin{array}{l}\text{The general solution} \\ y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|\end{array}}$$

$$y'' + y = \tan x$$