September 29 Math 3260 sec. 53 Fall 2025

3.8 Matrix Equations

We'll consider two types of matrix equations.

Matrix-Vector Equation

$$A\vec{x} = \vec{y}$$

The matrix A and the vector $\vec{y}$ are known. The variable to be solved for is the vector $\vec{x}$.

Matrix-Matrix Equation

$$AX = Y$$

The matrices A and Y are known. The variable to be solved for is the matrix X.

Matrix-Vector Equation

Suppose $A = [a_{ij}]$ is an $m \times n$ matrix, and let $\vec{y} = \langle y_1, y_2, \ldots, y_m \rangle$ be a vector in R^m.

$$A\vec{x} = \vec{y} \tag{1}$$

- Any vector $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in R^n that satisfies equation (1) is called a **solution**.
- The **solution set** of equation (1) is the set of all vectors $\vec{x}$ in R^n that are solutions.
- Equation (1) may have no solution, exactly one solution, or infinitely many solutions.

$A\vec{x} = \vec{y}$

The vector $\vec{x}$ is a solution of the matrix-vector equation $A\vec{x} = \vec{y}$ if and only if $(x_1, x_2, \ldots, x_n)$ is a solution of the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= y_m. \end{aligned}$$

Note this is the system having coefficient matrix A and augmented matrix $\widehat{A} = \left[A \,|\, \vec{y}\right]$.

Theorem

Suppose that A is an $m \times n$ matrix and that $\vec{y}$ is a vector in R^m and consider the matrix–vector equation

$$A\vec{x} = \vec{y}.$$

Let $\widehat{A}$ be the augmented matrix $\widehat{A} = \left[\begin{array}{c|c} A & \vec{y} \end{array}\right]$.

1. If the rightmost column of $\widehat{A}$ is a pivot column of $\widehat{A}$, then $A\vec{x} = \vec{y}$ is inconsistent.
2. If the rightmost column of $\widehat{A}$ is not a pivot column of $\widehat{A}$, then $A\vec{x} = \vec{y}$ is consistent.

Moreover, if $A\vec{x} = \vec{y}$ is consistent then

1. If every column of A is a pivot column of A, then $A\vec{x} = \vec{y}$ has a unique solution.
2. If at least one column of A is not a pivot column of A, then $A\vec{x} = \vec{y}$ has infinitely many solutions.

Example: Consider $A\vec{x} = \vec{y}$ for A and $\vec{y}$ given.

$$\text{(a)} \quad A = \begin{bmatrix} 3 & 3 \\ 4 & -2 \end{bmatrix}, \quad \text{and} \quad \vec{y} = \langle 5, 2 \rangle$$

$A \vec{x}$

2×2 $\quad R^2$

$\downarrow$

R^2

- If $\vec{x} \in R^n$, what is n? $n = 2$
- Is $A\vec{x} = \vec{y}$ consistent or inconsistent?

$$[A \mid \vec{y}] = \left[\begin{array}{cc|c} 3 & 3 & 5 \\ 4 & -2 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 8/9 \\ 0 & 1 & 7/9 \end{array}\right]$$

not a pivot column

The equation is consistent.

Example: Consider $A\vec{x} = \vec{y}$ for A and $\vec{y}$ given.

$$\text{(b)} \quad A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 3 & 4 \end{bmatrix}, \quad \text{and} \quad \vec{y} = \langle 5, 2, 0 \rangle$$

$A\vec{x}$

$3\times 2 \quad R^2 \downarrow R^3$

- If $\vec{x} \in R^n$, what is n? $n = 2$
- Is $A\vec{x} = \vec{y}$ consistent or inconsistent?

$$[A \mid \vec{b}] = \left[\begin{array}{cc|c} 2 & 1 & 5 \\ -1 & -2 & 2 \\ 3 & 4 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

not a pivot column

The equation is consistent

Example: Consider $A\vec{x} = \vec{y}$ for A and $\vec{y}$ given.

$$\text{(c)} \quad A = \begin{bmatrix} 2 & 1 \\ -1 & -2 \\ 3 & 4 \end{bmatrix}, \quad \text{and} \quad \vec{y} = \langle 2, 1, 2 \rangle$$

- If $\vec{x} \in R^n$, what is n? $n = 2$
- Is $A\vec{x} = \vec{y}$ consistent or inconsistent?

$$[A \,|\, \vec{y}] = \left[\begin{array}{cc|c} 2 & 1 & 2 \\ -1 & -2 & 1 \\ 3 & 4 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

pivot column

The equation is inconsistent.

Example

Find the solution set of the equation $A\vec{x} = \vec{y}$ where $A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & -3 & 0 \end{bmatrix}$ and $\vec{y} = \langle 1, -1 \rangle$.

$A\vec{x}$

2×3 $\quad \mathbb{R}^3 \downarrow \mathbb{R}^2$

$[A \mid \vec{y}] = \left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 4 & -3 & 0 & -1 \end{array}\right] \quad -2R_1 + R_2 \to R_2$

$\begin{array}{cccc} -4 & -2 & -4 & -2 \\ 4 & -3 & 0 & -1 \end{array}$

$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & -5 & -4 & -3 \end{array}\right] \quad -\frac{1}{5}R_2 \to R_2$

$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 1 & \frac{4}{5} & \frac{3}{5} \end{array}\right] \quad -R_2 + R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 2 & 0 & \frac{6}{5} & \frac{2}{5} \\ 0 & 1 & \frac{4}{5} & \frac{3}{5} \end{array}\right]$$

$$\begin{array}{cccc} 0 & -1 & -\frac{4}{5} & -\frac{3}{5} \\ 2 & 1 & \frac{10}{5} & \frac{5}{5} \end{array}$$

$\frac{1}{2}R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & \frac{4}{5} & \frac{3}{5} \end{array}\right]$$

$\vec{x} = \langle x_1, x_2, x_3 \rangle$,

$x_1 = \frac{1}{5} - \frac{3}{5}x_3$

$x_2 = \frac{3}{5} - \frac{4}{5}x_3$

x_3 is free

A solution

$$\vec{x} = \left\langle \frac{1}{5} - \frac{3}{5}x_3, \frac{3}{5} - \frac{4}{5}x_3, x_3 \right\rangle$$

Let $x_3 = t$.

$$\vec{x} = \left\langle \frac{1}{5}, \frac{3}{5}, 0 \right\rangle + t \left\langle \frac{-3}{5}, \frac{-4}{5}, 1 \right\rangle, \quad t \in \mathbb{R}.$$

We want to keep in mind that since the equation was presented as a matrix-vector equation, the solution should be given in the vector format (not just the parametric--list--format).