September 2 Math 2306 sec. 51 Fall 2022

Section 5: First Order Equations Models and Applications

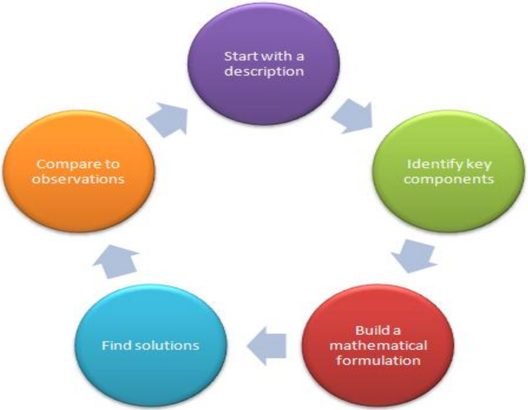


Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2021, there were 58 rabbits. In 2022, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2031.

We need variables. The population is changing in time, so let's introduce variables

$$t \sim \text{time} \quad \text{and} \quad P(t) \sim \text{is the population (density) at time } t.$$

We need to express the following mathematically:

The population's rate of change is proportional to the population.

$$\frac{dP}{dt} \propto P \quad \Longrightarrow \quad \frac{dP}{dt} = kP$$

rate of change

proportional

for constant k.

If we take t in years, with $t=0$ in 2021. The given info gives

$$P(0) = 58 \quad \text{and} \quad P(1) = 89$$

We have an IVP

$$\frac{dP}{dt} = kP, \quad P(0) = 58$$

with an extra condition, $P(1) = 89$.

The ODE is separable and linear.

Separating the variables

$$\frac{1}{P}\frac{dP}{dt} = k \implies \frac{1}{P}\,dP = k\,dt$$

$$\int \frac{1}{P}\,dP = \int k\,dt \Rightarrow \ln P = kt + C$$

Solving for P

$$P = e^{kt+C} = e^{kt} \cdot e^{C} = A e^{kt}$$

where $A = e^{C}$. Apply $P(0) = 58$

$$P(0) = A e^{0} = 58 \Rightarrow A = 58$$

The population $P(t) = 58 e^{kt}$

To find k, use $P(1) = 89$.

$$P(1) = 58 e^{k} = 89$$

Solving for k

$$e^k = \frac{89}{58} \implies k = \ln\left(\frac{89}{58}\right)$$

The population

$$\boxed{P(t) = 58\, e^{t \ln\left(\frac{89}{58}\right)}}$$

with t in years since 2021, $t=10$ in 2031. This model predicts

$$P(10) = 58\, e^{10 \ln\left(\frac{89}{58}\right)} \approx 4200$$

About 4200 rabbits are expected in the population in 2031.

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Decay is usually written

$$\frac{dP}{dt} = -kP \quad \text{w/} \quad k > 0.$$

Series Circuits: RC-circuit

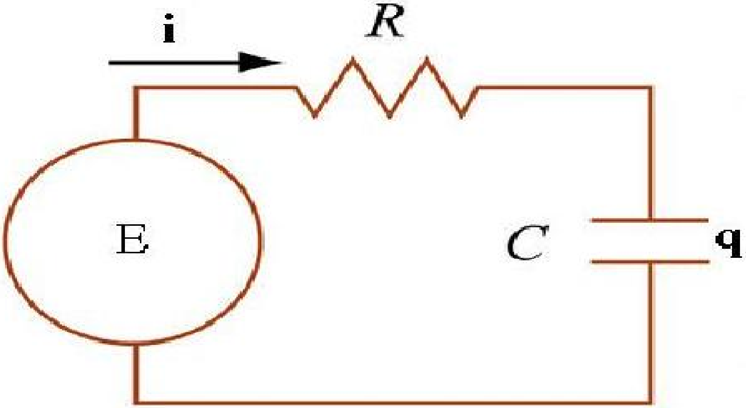


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

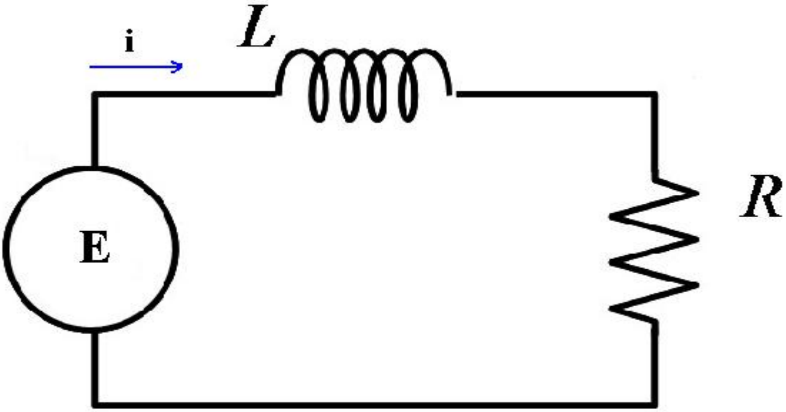


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω),
Inductance L in henries (h),
Capacitance C in farads (f),
Implied voltage E in volts (V),
Charge q in coulombs (C),
Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

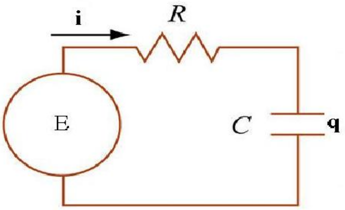
Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

RC Series Circuit

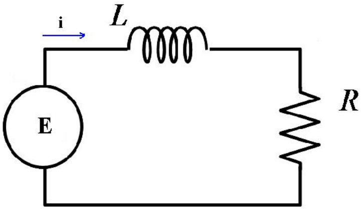


drop across resistor	$+$	drop across capacitor	$=$	applied force
$R\frac{dq}{dt}$	$+$	$\frac{1}{C}q$	$=$	$E(t)$

$$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t)}$$

If $q(0) = q_0$, the IVP can be solved to find $q(t)$ for all $t > 0$.

LR Series Circuit



$$\begin{array}{ccccc} \text{drop across inductor} & + & \text{drop across resistor} & = & \text{applied force} \\ L\frac{di}{dt} & + & Ri & = & E(t) \end{array}$$

$$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

If $i(0) = i_0$, the IVP can be solved to find $i(t)$ for all $t > 0$.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

General equation

$$R\frac{dq}{dt} + \frac{1}{C}q = E \qquad i(0) = q'(0) = 0.4$$

$$E(t) = 200, \quad R = 1000, \quad C = 5 \cdot 10^{-6}$$

$$1000\, q' + \frac{1}{5 \cdot 10^{-6}} q = 200$$

$$* \quad \frac{1}{5 \cdot 10^{-6}} \div 1000 = \frac{10^6}{5(10^3)} = \frac{10^3}{5} = 200$$

In standard form

$$q' + 200q = \frac{1}{5}$$

$$P(t) = 200, \quad \mu = e^{\int P(t)\,dt} = e^{\int 200\,dt} = e^{200t}$$

$$\frac{d}{dt}\left(e^{200t} q\right) = \frac{1}{5} e^{200t}$$

$$e^{200t} q = \int \frac{1}{5} e^{200t}\,dt$$

$$= \frac{1}{1000} e^{200t} + k$$

The general solution

$$q = \frac{1}{1000} + k e^{-200t}$$

Apply $q'(0) = 0.4 = \frac{2}{5}$

$$q' = -200 k e^{-200t}$$

$$q'(0) = -200 k e^{0} = \frac{2}{5}$$

$$\Rightarrow \quad k = \frac{2}{5}\left(\frac{-1}{200}\right)$$

$$= \frac{-1}{500}$$

The charge

$$q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}$$

$$\lim_{t\to\infty} q(t) = \frac{1}{1000}$$

The long time charge is expected to be $\frac{1}{1000}$ C