September 30 Math 2306 sec. 52 Fall 2022

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Basics of the Method

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

- Confirm the left is constant coefficient.
- Identify what **type** of function g is.
- Assume y_p is the same **type.****
- Remember that polynomials include all decending powers, and sines and cosines go together.
- The principle of superposition can be used if the right side is a sum of different kinds of g's.

**This may need some modification depending on the homogeneous equation as we'll see shortly

The Superposition Principle

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

The principle of superposition for nonhomogeneous equations tells us that we can find y_p by considering separate problems

$$y_{p_1} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x)$$

$$y_{p_2} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_2(x),$$

and so forth.

Then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$.

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

The principle of superposition says that we can consider the problem in parts.

1) Find y_{p_1} solving $y'' - 4y' + 4y = 6e^{-3x}$ $\boxed{y_{p_1} = Ae^{-3x}}$

2) Find y_{p_2} solving $y'' - 4y' + 4y = 16x^2$ $\boxed{y_{p_2} = Bx^2 + Cx + D}$

$$y_p = y_{p_1} + y_{p_2}$$

A Glitch!

What happens if the assumed form for y_p is part[1] of y_c? Consider applying the process to find a particular solution to the ODE

$$y'' - 2y' = 3e^{2x}$$

$g(x) = 3e^{2x}$ a constant times e^{2x}

Set $y_p = Ae^{2x}$ sub.

$y_p' = 2Ae^{2x}$

$y_p'' = 4Ae^{2x}$

$$y_p'' - 2y_p' = 3e^{2x}$$

$$4Ae^{2x} - 2(2Ae^{2x}) = 3e^{2x}$$

$$0 = 3e^{2x}$$

[1] A term in $g(x)$ is contained in a fundamental solution set of the associated homogeneous equation.

This is false for any choice of A.

Our guess for y_p is part of y_c.

We can try to salvage the method by including a factor of x.

$$y'' - 2y' = 3e^{2x}$$

Set $y_p = (Ae^{2x})x$

$y_p = Axe^{2x}$ sub

$y_p' = Ae^{2x} + 2Axe^{2x}$

$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$

$= 4Ae^{2x} + 4Axe^{2x}$

$y_p'' - 2y_p' = 3e^{2x}$

$$4Ae^{2x} + 4Axe^{2x} - 2\left(Ae^{2x} + 2Axe^{2x}\right) = 3e^{2x}$$

Collect like terms

$$xe^{2x}\underbrace{(4A - 4A)}_{0} + e^{2x}(4A - 2A) = 3e^{2x}$$

$$2Ae^{2x} = 3e^{2x}$$

$$\Rightarrow A = \frac{3}{2}$$

$$\boxed{\text{Hence } y_p = \frac{3}{2}xe^{2x}}$$

$$y'' - 2y' = 3e^{2x}$$

Let's find y_c

y_c solves $y'' - 2y' = 0$

The characteristic eqn is

$$m^2 - 2m = 0 \implies m(m-2) = 0$$

$m = 0$ or $m = 2$ two real roots

$$y_1 = e^{0x} = 1, \quad y_2 = e^{2x}$$

So $y_c = C_1 + C_2 e^{2x}$

The general solution is $y = C_1 + C_2 e^{2x} + \frac{3}{2} x e^{2x}$

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case I: The guess for y_{p_i} **DOES NOT** have any like terms in common with y_c.

Then our guess for y_{p_i} will work as written. We do the substitution to find the A, B, etc.

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case II: The guess for y_{p_i} **DOES** have a like term in common with y_c.

Then we multiply our guess at y_{p_i} by x^n where n is the smallest positive integer such that our new guess $x^n y_{p_i}$ does not have any like terms in common with y_c. Then we take this new guess and substitute to find the A, B, etc.

Case II Examples

Find the general solution of the ODE.

$$y'' - 2y' + y = -4e^x$$

Find y_c: y_c solves $y'' - 2y' + y = 0$

Characteristic eqn: $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ double root

$y_1 = e^{1x}$, $y_2 = xe^x$

$y_c = c_1 e^x + c_2 x e^x$

$y'' - 2y' + y = -4e^x$

Find y_p: $g(x) = -4e^x$ const. times e^x

$y_p = Ae^x$ nope!

$y_p = (Ae^x)x = Axe^x$ nope!!

$y_p = (Ae^x)x^2 = Ax^2e^x$ yep!

$y_p = Ax^2e^x$ sub.

$y_p' = 2Axe^x + Ax^2e^x$

$y_p'' = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x$

$= 2Ae^x + 4Axe^x + Ax^2e^x$

$$y_p'' - 2y_p' + y_p = -4e^x$$

$$2Ae^x + 4Axe^x + Ax^2e^x - 2(2Axe^x + Ax^2e^x) + Ax^2e^x = -4e^x$$

Collect like terms e^x, xe^x, x^2e^x

$$x^2e^x(\underbrace{A-2A+A}_{0}) + xe^x(\underbrace{4A-4A}_{0}) + e^x(2A) = -4e^x$$

$$2Ae^x = -4e^x$$

$$\Longrightarrow \quad A = -2$$

Hence $y_p = -2x^2e^x$

From before $y_c = c_1e^x + c_2xe^x$

The general solution is

$$y = c_1 e^x + c_2 x e^x - 2x^2 e^x$$