September 3 Math 2306 sec. 51 Fall 2021

Section 5: First Order Equations Models and Applications

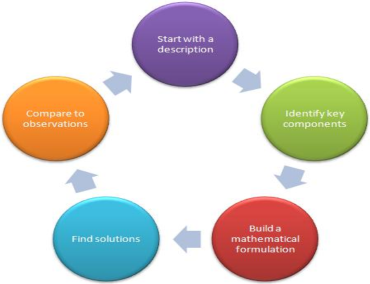


Figure: We've seen exponential growth/decay and simple linear circuits (RC or LR)

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

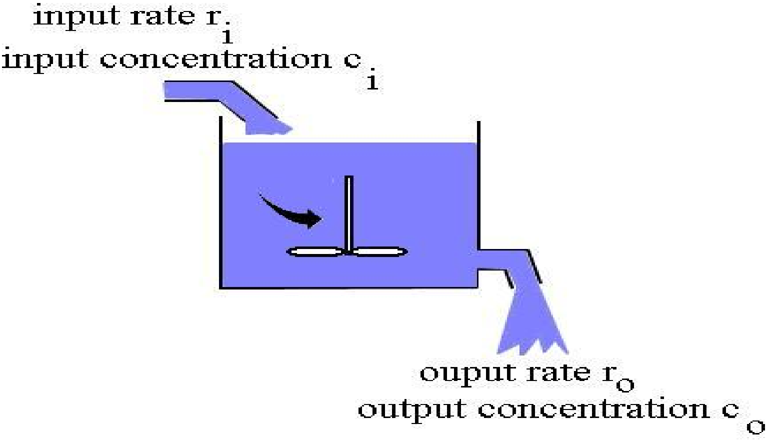


Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array}\right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

$$c_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$r_i c_i - r_o c_o$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

Fluid flow rates

$$r_i = 5 \ \frac{\text{gal}}{\text{min}}$$

$$r_o = 5 \ \frac{\text{gal}}{\text{min}}$$

Concentrations

$$c_i = 2 \ \frac{\text{lb}}{\text{gal}}$$

Since $r_i = r_o$, V is constant

$$V(t) = V(0) = 500 \text{ gal}$$

So $$C_o = \frac{A}{V} = \frac{A}{500} \ \frac{\text{lb}}{\text{gal}}$$

$$\frac{dA}{dt} = r_i c_i - r_o c_o$$

$$\frac{dA}{dt} = 5 \cdot 2 - 5 \frac{A}{500}$$

starts with pure water

$$\frac{dA}{dt} + \frac{1}{100} A = 10 \quad , \quad A(0) = 0$$

Here $P(t) = \frac{1}{100}$, so $\mu = e^{\int P(t)\,dt} = e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100}t}$

$$e^{\frac{1}{100}t}\left(A' + \frac{1}{100}A\right) = e^{\frac{1}{100}t}(10)$$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) = 10\, e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) dt = \int 10\, e^{\frac{1}{100}t}\, dt$$

$$e^{\frac{1}{100}t} A = 1000\, e^{\frac{1}{100}t} + C$$

The general solution

$$A = 1000 + C e^{\frac{-1}{100}t}$$

We'll apply $A(0) = 0$

$$0 = 1000 + C e^{0} \Rightarrow C = -1000$$

The amount of salt @ time t

$$\boxed{A(t) = 1000 - 1000\, e^{\frac{-1}{100}t}.}$$

After 5 minutes, the concentration $C(5)$ in the tank

$$C(5) = \frac{A(5)}{V(5)} = \frac{1000 - 1000\, e^{\frac{-1}{100}(5)}\ \text{lb}}{500\ \text{gal}}$$

$$= 2 - 2e^{\frac{-1}{20}}\ \frac{\text{lb}}{\text{gal}}$$

$$\approx 0.0975\ \frac{\text{lb}}{\text{gal}}$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

$r_i = 5 \frac{\text{gal}}{\text{min}}$ $\quad C_i = 2 \frac{\text{lb}}{\text{gal}}$

$r_o = 10 \frac{\text{gal}}{\text{min}}$ $\quad$ The Volume

$V(t) = V(0) + r_i t - r_o t$

$= 500 - 5t$

so $C_o = \frac{A}{V} = \frac{A}{500-5t}$ $\qquad \frac{dA}{dt} = r_i C_i - r_o C_o$

$\frac{dA}{dt} + 10\frac{A}{500-5t} = 10 \Rightarrow \boxed{\frac{dA}{dt} + \frac{2}{100-t}A = 10}$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

rate of change of P is $\frac{dP}{dt}$. It's proportional to the product of P and $M-P$.

$$\frac{dP}{dt} = kP(M-P)$$

for some constant of proportionality k.

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M - P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

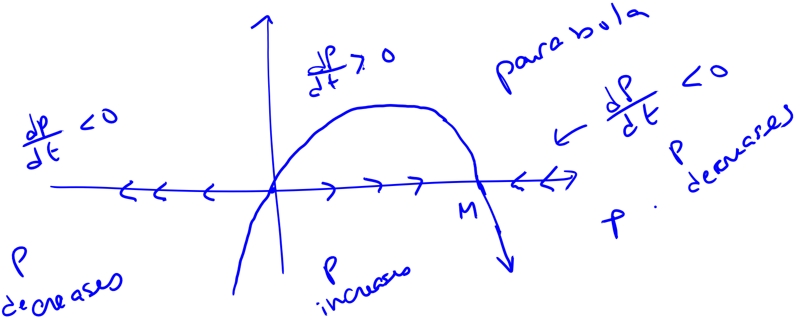
$$\frac{dP}{dt} = kMP - kP^2$$

$$\Rightarrow \quad \frac{dP}{dt} - kMP = -kP^2$$

It's Bernoulli w/ $n=2$

$$\frac{dP}{dt} = kP(M-P)$$

$k>0$, $M>0$. $h(P) = P(M-P)$



For any starting $P \neq 0$, P tends toward M.