September 3 Math 2306 sec. 52 Fall 2021

Section 5: First Order Equations Models and Applications

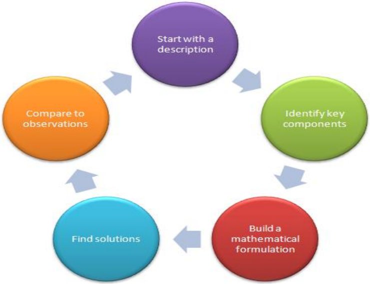


Figure: We've seen exponential growth/decay and simple linear circuits (RC or LR)

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

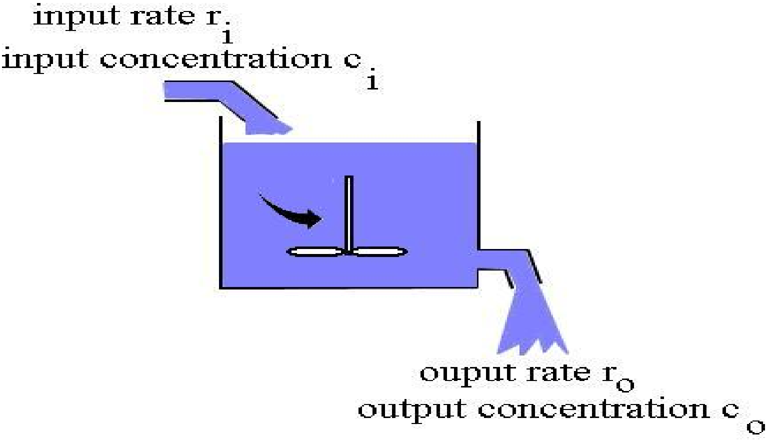


Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array}\right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

$$c_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0)+(r_i-r_o)t}.$$

$$r_i c_i - r_o c_o$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

$$\frac{dy}{dt} + P(t)\, y = f(t)$$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

fluid flow rates:

$$r_i = 5\ \frac{\text{gal}}{\text{min}}$$

$$r_o = 5\ \frac{\text{gal}}{\text{min}}$$

Concentrations in in/out flow:

$$c_i = 2\ \frac{\text{lb}}{\text{gal}}$$

$$c_o = \frac{A}{V}\ \frac{\text{lb}}{\text{gal}} = \frac{A}{500}\ \frac{\text{lb}}{\text{gal}}$$

$$V(t) = V(0) + (r_i - r_o)\,t = 500 + (5-5)t$$

$V(t) = 500$ gal for all t

$$\frac{dA}{dt} = r_i c_i - r_o c_o$$

$$= 5(2) - 5\,\frac{A}{500}$$

pure water to start

$$\frac{dA}{dt} + \frac{1}{100}A = 10 \qquad A(0) = 0$$

$$P(t) = \frac{1}{100} \Rightarrow \mu = e^{\int P(t)\,dt} = e^{\int \frac{1}{100}dt} = e^{\frac{1}{100}t}$$

$$e^{\frac{1}{100}t}\left(\frac{dA}{dt} + \frac{1}{100}A\right) = e^{\frac{1}{100}t}(10)$$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t}A\right) = 10\,e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) dt = \int 10\, e^{\frac{1}{100}t}\, dt$$

$$e^{\frac{1}{100}t} A = 1000\, e^{\frac{1}{100}t} + C$$

The general solution

$$A = 1000 + C e^{-\frac{1}{100}t}$$

Using the condition $A(0) = 0$

$$0 = 1000 + C e^{0} \Rightarrow C = -1000$$

The amount of salt

$$\boxed{A(t) = 1000 - 1000\, e^{-\frac{1}{100}t}}$$

After 5 minutes, the concentration in the tank is

$$\frac{A(5)}{V(5)} = \frac{1000 - 1000\, e^{\frac{-1}{100}(5)} \text{ lb}}{500 \text{ gal}}$$

$$= 2 - 2e^{\frac{-1}{20}} \approx 0.0975 \text{ lb/gal}$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

$r_i = 5 \frac{\text{gal}}{\text{min}}$ $\quad c_i = 2 \frac{\text{lb}}{\text{gal}}$

$r_o = 10 \frac{\text{gal}}{\text{min}}$ $\quad$ Volume $V = 500 + 5t - 10t$

$= 500 - 5t$

$\frac{dA}{dt} = r_i c_i - r_o c_o \qquad c_o = \frac{A}{V} = \frac{A}{500-5t}$

$\frac{dA}{dt} = 5(2) - 10\frac{A}{500-5t} \implies \boxed{\frac{dA}{dt} + \frac{2}{100-t}A = 10}$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

The rate of change of P is $\frac{dP}{dt}$. It's proportional to the product of P and $M-P$.

Hence $\frac{dP}{dt} = kP(M-P)$ for some constant of proportionality k.

[1]The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**
Solve this equation and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

Separable $\frac{dP}{dt} = g(t)h(P)$ where

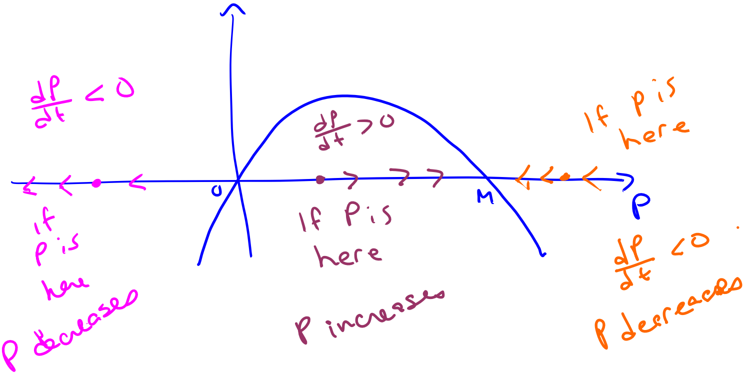
$g(t) = k$ and $h(P) = P(M-P)$

Note $\frac{dP}{dt} = kMP - kP^2$

$\frac{dP}{dt} - kMP = -kP^2$ Bernoulli with $n=2$.

$$\frac{dP}{dt} = kP(M-P) \qquad k>0$$

Let $h(P) = P(M-P)$ plot h



If P starts anywhere to the right of zero, it wants to tend to M.