September 3 Math 2306 sec. 54 Fall 2021

Section 5: First Order Equations Models and Applications

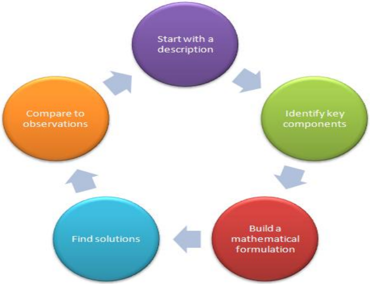


Figure: We've seen exponential growth/decay and simple linear circuits (RC or LR)

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

The model for the RC circuit is

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

The ODE for this problem ended up in standard for as

$$\frac{dq}{dt} + 200q = \frac{1}{5} \quad \text{with the condition} \quad q'(0) = \frac{2}{5}.$$

Here $P(t) = 200$, $\mu = e^{\int P(t)dt} = e^{\int 200dt} = e^{200t}$

$$e^{200t}(q' + 200q) = \frac{1}{5}e^{200t}$$

$$\frac{d}{dt}\left(e^{200t} q\right) = \frac{1}{5} e^{200t}$$

$$\int \frac{d}{dt}\left(e^{200t} \cdot q\right) dt = \int \frac{1}{5} e^{200t}\, dt$$

$$e^{200t} q = \frac{1}{1000} e^{200t} + k$$

Hence $q(t) = \frac{1}{1000} + k e^{-200t}$

we'll find k using $q'(0) = \frac{2}{5}$

$$q'(t) = -200 k e^{-200t}$$

$$q'(0) = -200 k e^{0} = \frac{2}{5}$$

$$k = \frac{2}{5(-200)} = \frac{-1}{500}$$

$$\boxed{\text{The charge on the capacitor} \quad q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}}$$

$$\lim_{t \to \infty} q(t) = \frac{1}{1000} \text{ Coulombs}$$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

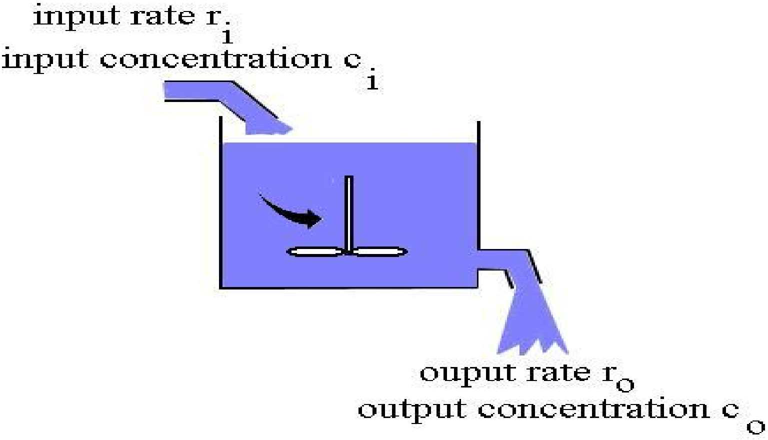


Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c}\textit{input rate}\\ \textit{of salt}\end{array}\right) - \left(\begin{array}{c}\textit{output rate}\\ \textit{of salt}\end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

$$c_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0)+(r_i-r_o)t}.$$

$$r_i c_i \quad - r_o c_o$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

fluid flow rates

$r_i = 5\ \frac{\text{gal}}{\text{min}}$

$r_o = 5\ \frac{\text{gal}}{\text{min}}$

Concentrations

$c_i = 2\ \frac{\text{lb}}{\text{gal}}$

$c_o = \frac{A}{V}\ \frac{\text{lb}}{\text{gal}} = \frac{A}{500}\ \frac{\text{lb}}{\text{gal}}$

$V(t) = V(0) + r_i t - r_o t = 500 + 5t - 5t = 500$

$\frac{dA}{dt} = r_i c_i - r_o c_o$

$$\frac{dA}{dt} = 5(2) - 5\frac{A}{500}$$

pure water has no salt

$$\frac{dA}{dt} + \frac{1}{100}A = 10 \qquad A(0) = 0$$

$$P(t) = \frac{1}{100}, \quad \mu = e^{\int P(t)\,dt} = e^{\int \frac{1}{100}dt} = e^{\frac{1}{100}t}$$

$$e^{\frac{1}{100}t}\left(A' + \frac{1}{100}A\right) = 10e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t}A\right) = 10e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{100}t}A\right)dt = \int 10e^{\frac{1}{100}t}\,dt$$

$$e^{\frac{1}{100}t} A = 1000\, e^{\frac{1}{100}t} + C$$

Hence $\quad A(t) = 1000 + C e^{-\frac{1}{100}t}$

Applying the I.C. $\quad A(0) = 1000 + C e^{0} = 0$

$$C = -1000$$

$$\boxed{\begin{array}{l}\text{The amount of salt is} \\ \qquad A(t) = 1000 - 1000\, e^{-\frac{1}{100}t}\end{array}}$$

The concentration after 5 minutes is

$$\frac{A(5)}{V} = \frac{1000 - 1000e^{\frac{-1}{100}(5)}}{500} \ \frac{\text{lb}}{\text{gal}}$$

$$= 2 - 2e^{\frac{-1}{20}} \approx 0.0975$$

$r_i \neq r_o$

$$\frac{dA}{dt} = r_i c_i - r_o c_o$$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

Now, $r_i = 5$ and $r_o = 10$ so

$$V(t) = 500 + (5 - 10)t = 500 - 5t$$

So $\quad c_o = \frac{A}{V} = \frac{A}{500-5t}$

$$\frac{dA}{dt} = 5(2) - 10\,\frac{A}{500-5t}$$

In standard form

$$\frac{dA}{dt} + \frac{2}{100-t}A = 10$$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

The rate of change of P, $\frac{dP}{dt}$, is proportional to the product of P and $M-P$.

$$\frac{dP}{dt} = kP(M-P)$$

for some constant of proportionality k.

[1]The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M - P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

Separable $\quad \frac{dP}{dt} = g(t)\,h(P)$

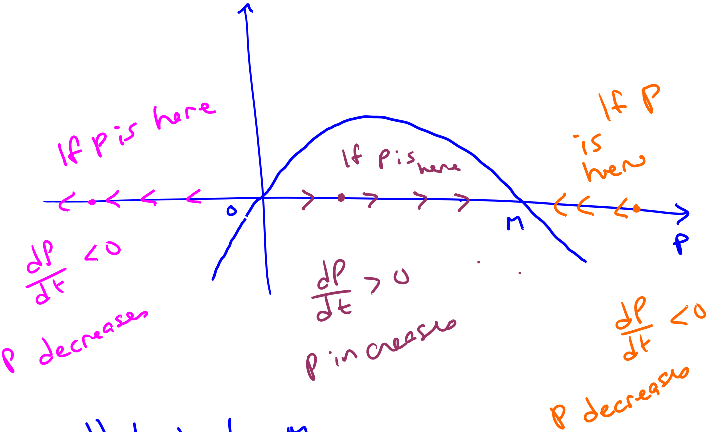
$$g(t) = k \quad , h(P) = P(M-P)$$

Bernoulli $\quad \frac{dP}{dt} = kMP - kP^2$

$$\frac{dP}{dt} - kMP = -kP^2 \qquad n = 2$$

$$\frac{dP}{dt} = k\,P(M-P) \qquad k>0\ ,\ M>0$$

Let $h(P) = P(M-P)$



If $P(0) > 0$; P should tend to M.