September 3 Math 3260 sec. 53 Fall 2025

Chapter 2 Systems of Linear Equations

In this chapter, we will consider equations that have a special structure known as **linearity**. We will

- define linear equations and linear systems,
- define solutions and solution sets,
- learn Gaussian elimination,
- introduce matrices as a tool for solving linear systems,
- and learn how matrices can be used to solve linear systems.

Linear Equation & System of Linear Equations

A **linear equation** in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers (scalars). The numbers $a_1, \ldots, a_n$ are called the **coefficients**, and b can be called the **constant term**.

A **system of linear equations** (a.k.a. a *linear system*) is a collection of one or more linear equations in the same variables considered together. A generic system of m equations in n variables is shown in equation (3).

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}.$$

Linear -vs- Not Linear

For an equation to be linear, only two operations can be done on the variables:

- multiply by coefficients (scalars), and
- add.

More exotic operations (powers, exponential, trig, etc.) aren't *allowed*.

Linear: $2x_1 - \sqrt{2}\, x_2 + 21x_3 = -1, \quad -3x_1 + 4x_2 = 0$

Not Linear: $x_1 x_2 + x_3 = 4, \quad x_1^3 - e^{x_2} = 0$

Examples

An example of a system of two equations in four variables.

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array} \qquad (1)$$

An example of a system of three equations in three variables.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & + & 3x_2 & - & 2x_3 & = & 0 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array} \qquad (2)$$

Homogeneous -vs- Nonhomogeneous

A system is called **homogeneous** if all constant terms are zero. Otherwise, it's called **nonhomogeneous**.

The system (1) above is nonhomogeneous.

The system (2) above is homogeneous.

Solutions & Solution Sets

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}. \tag{3}$$

Solutions & Solution Sets

A **solution** of (3) is as an ordered n-tuple of real numbers, $(s_1, s_2, \ldots, s_n)$, having the property that upon substitution,

$$x_1 = s_1, \quad x_2 = s_2, \quad \cdots, \quad x_n = s_n,$$

every equation in the system reduces to an identity. The collection of all solutions of (3) is called the **solution set** of the system.

Example

Show that $(1,2,3)$ is a solution of the homogeneous system

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & + & 3x_2 & - & 2x_3 & = & 0 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array}.$$

Example

The solution set of this system is $\{(t, 2t, 3t) \mid t \in R\}$.

$$\begin{array}{ccccccc} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & + & 3x_2 & - & 2x_3 & = & 0 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array}.$$

Expressing Solutions

There are several ways that we can write solutions to a system of equations.

- As a point $(s_1, s_2, \dots, s_n)$—e.g., $(1, 2, 3)$.
- Set builder notation—e.g., $\{(t, 2t, 3t) \mid t \in R\}$

Parametric

A **parametric description**—or **parametric form**—is a list.

$$\begin{array}{rcl} x_1 & = & t \\ x_2 & = & 2t \\ x_3 & = & 3t \end{array} \quad -\infty < t < \infty \;, \qquad \text{or} \qquad \begin{array}{rcl} x_1 & = & 1 \\ x_2 & = & 2 \\ x_3 & = & 3 \end{array}$$

Vector Parametric

A **vector parametric description**—or **vector parametric form**—is a solution/solutions expressed as a vector/vectors.

$$\vec{x} = \langle 1, 2, 3 \rangle, \qquad \text{or} \quad \vec{x} = t\langle 1, 2, 3 \rangle, \quad t \in R.$$

Example

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array} \qquad (4)$$

It can be shown that the solutions of (4) are the 4-tuples (x_1, x_2, x_3, x_4) that satisfy

$$x_1 = 5 + 2x_3 - 4x_4, \quad \text{and} \quad x_2 = -3x_3 + x_4$$

where x_3 and x_4 can be any real number. A **parametric description** could be written

$$\begin{array}{rcl} x_1 & = & 5 + 2s - 4t, \\ x_2 & = & -3s + t, \\ x_3 & = & s, \\ x_4 & = & t, \end{array} \quad , \quad s, t \in R.$$

Remark: $s, t \in R$, read "*s and t in R*" is shorthand for $-\infty < s < \infty$ and $-\infty < t < \infty$.

Parametric to Vector Parametric

$$\begin{array}{rcl} x_1 & = & 5+2s-4t, \\ x_2 & = & -3s+t, \\ x_3 & = & s, \\ x_4 & = & t, \end{array} \quad , \quad s,t \in R.$$

Convert this parametric description to vector parametric.

Existence & Uniqueness

Definition: Equivalent Systems

We will say that two systems of linear equations are **equivalent** if they have the same solution set.

Existence & Uniqueness

Theorem: For a system of linear equations, exactly one of the following is true:

i. the solution set is empty (i.e., there is no solution),

ii. there exists a unique solution, or

iii. there are infinitely many solutions.

If a system has at least one solution, we call it **consistent**. Otherwise, we call the system **inconsistent**.

Homogeneous Systems & the Trivial Solution

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}.$$

Consider a homogeneous sytem in n variables.

- Such a system is always consistent.
- The zero vector, $\vec{x} = \vec{0}_n$, is necessarily a solution. We call this **the trivial solution**.
- If the system has exactly one solution, then it must be the trivial solution.
- If the system has infinitely many solutions, we call any solution that is not the zero vector **a nontrivial solution**.

2.1.1 Systems of Two Equations with Two Variables

A system of two equations in two variables has the form

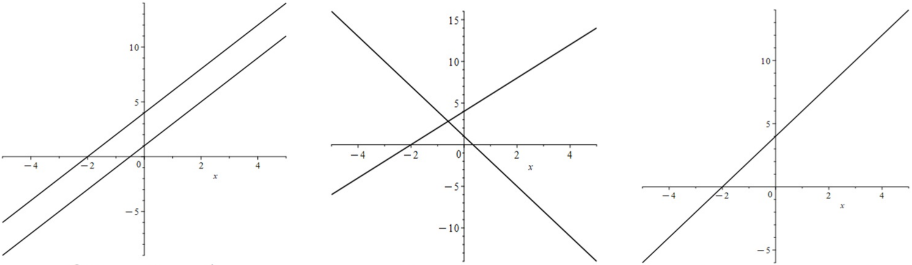
$$\begin{array}{ccccc} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array} .$$

We can think of the two equations as corresponding to a pair of lines, something like

$$x_2 = (\text{slope})\, x_1 + (\text{intercept}).$$

Such systems allow us to compare the three solution cases geometrically.

The three solution cases are easily visualized in R^2.



$$\begin{aligned} 2x_1 - x_2 &= 4 \\ 2x_1 - x_2 &= 1 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= 4 \\ 3x_1 + x_2 &= -1 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= 4 \\ -4x_1 + 2x_2 &= -8 \end{aligned}$$

Figure: Lines determined by two linear equations in two variables illustrating the three possible geometric relationships.

i. parallel, non-intersecting lines correspond to an inconsistent system,

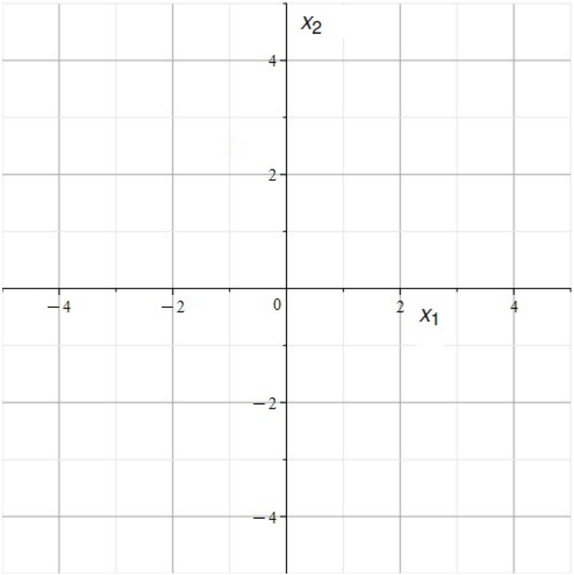
ii. lines with two distinct slopes correspond to a system with one solution,

iii. coincident lines corresponds to a system with infinitely many solutions.

Example

Translate each equation into a line and determine if the system is consistent or inconsistent. If consistent, state whether there is a unique solution or infinitely many solutions.

$$\begin{array}{rcrcr} 3x_1 & - & x_2 & = & 4 \\ 4x_1 & + & 2x_2 & = & -2 \end{array}$$

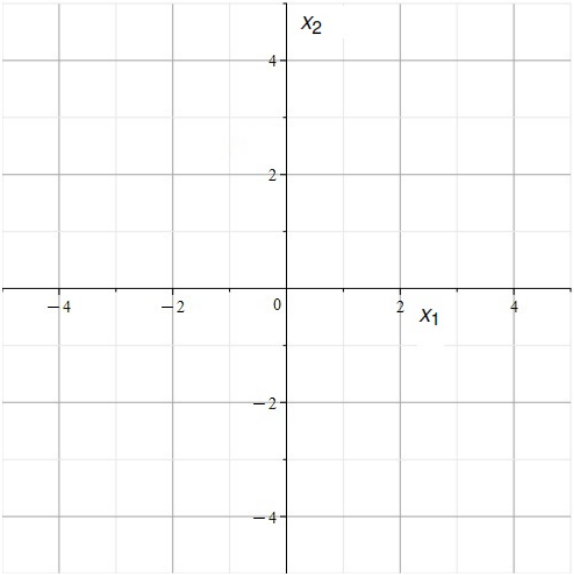


x_2
4
2
−4
−2
0
2
x_1
4
−2
−4

Example

Translate each equation into a line and determine if the system is consistent or inconsistent. If consistent, state whether there is a unique solution or infinitely many solutions.

$$\begin{array}{rcrcr} -5x_1 & + & 3x_2 & = & 6 \\ 2x_1 & - & \frac{6}{5}x_2 & = & -\frac{12}{5} \end{array}$$


x_2
4
2
−2
−4
−4
−2
0
2
4
x_1

3 Equations in 3 Variables

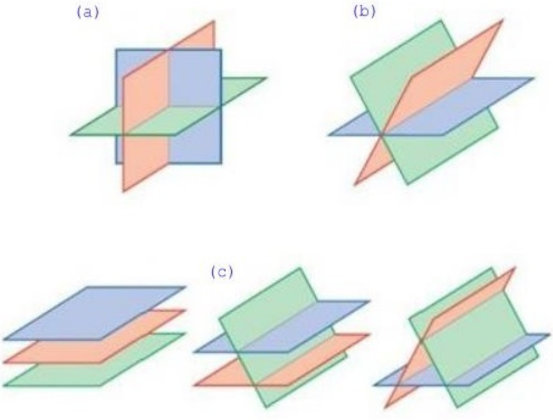


Figure: The graph of $a_1x_1 + a_2x_2 + a_3x_3 = b$ is a plane. Three may (a) intersect in a single point, (b) intersect in infinitely many points, or (c) not intersect in various ways.

2.2 Solving a System of Linear Equations

Consider the pair of systems

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 2 \\ 3x_1 & + & x_2 & - & x_3 & = & 2 \\ -x_1 & - & 3x_2 & & & = & -7 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 2 \\ & & x_2 & + & x_3 & = & 5 \\ & & & & x_3 & = & 3 \end{array}$$

These systems are **equivalent** (this is NOT obvious). We can get the solution to the system on the right by a little back substitution.

Operations on Systems

There are three operations we can perform on a system that results in an equivalent system.

Elementary Equation Operations

- Multiply an equation by a nonzero scalar k. (scale)
- Interchange the position of any two equations. (swap)
- Replace an equation with the sum of itself and a multiple of any other equation. (replace)

When we add two equations, we add **like terms**.

Some Notation

We'll use E_i to refer to the i^{th} equation at any step in the solution process. Symbolically, we can write each operation:

- (scale) Replace equation i with k times itself: $kE_i \to E_i$
- (swap) Interchange equations i and j: $E_i \leftrightarrow E_j$
- (replace) Replace equation j with the sum of k times equation i plus equaiton j: $kE_i + E_j \to E_j$

Scale

To indicate that we are scaling equation E_i by the nonzero factor k, we'll write

$$kE_i \to E_i$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{-2E_3 \to E_3} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ -2x_1 & - & 2x_2 & - & 2x_3 & = & -12 \end{array}$$

Swap

To indicate that we are swapping equations E_i and E_j, we'll write

$$E_i \leftrightarrow E_j$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{E_1 \leftrightarrow E_3} \quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \end{array}$$

Replace

To indicate that we are replacing equation E_j with the sum of itself and k times equation E_i, we'll write

$$kE_i + E_j \rightarrow E_j$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{-2E_1 + E_2 \rightarrow E_2} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & - & 4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Note

$$\begin{array}{lrcrcrcr} & -2x_1 & - & 4x_2 & + & 2x_3 & = & 8 \\ & 2x_1 & & & + & x_3 & = & 7 \\ \hline \text{(add)} & 0x_1 & - & 4x_2 & + & 3x_3 & = & 15 \end{array}$$

Gaussian Elimination

We'll use some sequence of the three equation operations. Our goal is to change a system that looks something like

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

into one that looks something like

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & x_2 & - & 2x_3 & = & -10 \\ & & & & x_3 & = & 5 \end{array}.$$

If we can do that, we can use back substitution to find x_2, then x_1 and identify the solution.

Example

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Other Solution Cases

An example of the no solution case: The system

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & x_2 & - & x_3 & = & 2 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 0 \end{array}$$

leads to

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 0 \\ & & & & 0 & = & 1 \end{array}$$

Inconsistent systems always give rise to an equation that is false.

$$0 = \text{something nonzero}$$

Other Solution Cases

An example of an infinite solutions case: The system

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & 7x_3 & = & 10 \\ 2x_1 & - & x_2 & - & 8x_3 & = & 10 \\ -2x_1 & + & 2x_2 & + & 10x_3 & = & -12 \end{array}$$

leads to

$$\begin{array}{rcrcrcr} x_1 & & & - & 3x_3 & = & 4 \\ & & x_2 & + & 2x_3 & = & -2 \\ & & & & 0 & = & 0 \end{array}$$

Consistent systems with infinitely many solutions are always the result when

1. there are no false statements, and
2. there are more variables than nontrivial equations.

2.3 Matrices

Matrix

A **matrix** (plural *matrices*) is a rectangular array of numbers of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Each number, a_{ij}, is called an **entry** or an **element** of the matrix. If the matrix has m rows and n columns, we say that the **size** or **dimension** of the matrix is "m by n" and write $m \times n$.

$$a_{ij}$$

row index $\nearrow$ $\nwarrow$ column index

Coefficient & Augmented Matrices

Given a system of linear equations with m equations and n variables,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array},$$

the **coefficient matrix** for the system is the $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Coefficient & Augmented Matrices

Given a system of linear equations with m equations and n variables,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array},$$

the **augmented matrix** for the system is the $m \times n + 1$ matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

Example

Write the coefficient and augmented matrices for each system.

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array}$$

Example

Write the coefficient and augmented matrices for each system.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & + & 3x_2 & - & 2x_3 & = & 0 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array}$$

Elementary Row Operations

We will use matrices to perform elimination without involving the symbols and variable names. We have three operations we can perform on a matrix. We'll use the notation

$$R_i$$

to refer to the i^{th} row of the matrix.

Elementary Row Operations

- Multiply row i by any nonzero constant k (**scale**), $kR_i \to R_i$.
- Interchange row i and row j (**swap**), $R_i \leftrightarrow R_j$.
- Replace row j with the sum of itself and k times row i (**replace**), $kR_i + R_j \to R_j$.

Row Equivalence

Definition: We will say that two matrices are **row equivalent** if one matrix can be obtained from the other by performing some sequence of elementary row operations.

Remark: So every time we do a row operation, the result is row equivalent matrix.

Theorem:

If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent.

Gaussian Elimination & Augmented Matrix Structure

If we look back at the systems we've seen, we may note an identifiable structure in the matrices.

A Consistent System (one solution)

From

$$\begin{array}{rcrcrcr} 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \qquad \left[\begin{array}{rrr|r} 2 & 0 & 1 & 7 \\ 1 & 2 & -1 & -4 \\ 1 & 1 & 1 & 6 \end{array}\right]$$

we obtained

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & x_2 & - & 2x_3 & = & -10 \\ & & & & x_3 & = & 5 \end{array}, \qquad \left[\begin{array}{rrr|r} 1 & 2 & -1 & -4 \\ 0 & 1 & -2 & 10 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Note a sort of *triangular/trapezoidal* shape. Each row has a nonzero term on the left of the bar between the coefficient and augment columns.

Gaussian Elimination & Augmented Matrix Structure

A Consistent System (infinitely many solutions)

From

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & 7x_3 & = & 10 \\ 2x_1 & - & x_2 & - & 8x_3 & = & 10 \\ -2x_1 & + & 2x_2 & + & 10x_3 & = & -12 \end{array} \qquad \left[\begin{array}{rrr|r} 3 & 1 & -7 & 10 \\ 2 & -1 & -8 & 10 \\ -2 & 2 & 10 & -12 \end{array}\right]$$

we obtained

$$\begin{array}{rcrcrcr} x_1 & & & - & 3x_3 & = & 4 \\ & & x_2 & + & 2x_3 & = & -2 \\ & & & & 0 & = & 0 \end{array}, \qquad \left[\begin{array}{rrr|r} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This also has a triangular/trapezoidal structure. There are fewer nonzero rows than there are columns to the left of the bar.

Gaussian Elimination & Augmented Matrix Structure

An Inconsistent System

From

$$\begin{array}{rcrcrcl} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & x_2 & - & x_3 & = & 2 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 0 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 4 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ -1 & 3 & 4 & 0 \end{array}\right]$$

we obtained

$$\begin{array}{rcrcrcl} x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 0 \\ & & & & 0 & = & 1 \end{array}, \qquad \left[\begin{array}{rrr|r} 1 & 4 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Again, there is a triangle/trapezoid shape. A row with the only nonzero entry on the right side of the bar,

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right],$$

alerts us to a false statement.

When reading the row of a matrix, from left to right, we will call the first nonzero entry a **leading entry**.

Definition: Echelon Forms

We will say that a matrix is in **row echelon form** (ref) if it satisfies the properties that

1. any row whose entries are all zeros is below all rows that contain a leading entry, and

2. the leading entry in every row is to the right of the leading entries in every row above it.

We will say that a matrix is in **reduced row echelon form** (rref) if, in addition to being in row echelon form,

3. the leading entry in each row is a 1 (called a “leading one”), and

4. each leading one is the only nonzero entry in its column.

Example

Classify each matrix as a **row echelon form (ref)**, a **reduced row echelon form (rref)**, or not an echelon form.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example

Classify each matrix as a **row echelon form (ref)**, a **reduced row echelon form (rref)**, or not an echelon form.

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 6 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Row Reduction Algorithm

We will perform row operations to reduce A to an ref, and then to an rref. This is a two stage process that we can approach methodically.

- **Forward:** We work from left to right, top down to obtain an ref.
- **Backward:** From an ref, we work from right to left, bottom up to clear nonzero entries above the leading entries.

$$A = \begin{bmatrix} 3 & 2 & 1 & 6 & 0 \\ 4 & 2 & 2 & 0 & -2 \\ 1 & 1 & 0 & 3 & -2 \\ 2 & 1 & 1 & 3 & 2 \end{bmatrix}$$

Uniqueness of an rref

Theorem:

A matrix A is row equivalent to exactly one reduced echelon form.

Remark: A matrix can be row equivalent to lots of different refs, but to only one RREF. So it makes sense to call it THE rref, and to write

$$\text{rref}(A).$$

Pivot Position & Pivot Column

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Identifying Pivot Positions and Columns

The following matrices are **row equivalent**. Identify the pivot positions and pivot columns of the matrix A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Complete Row Reduction isn't needed to find Pivots

The following three matrices are row equivalent. (Note, B is an ref but not an rref, and C is an rref.)

$$A = \begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Identify the pivot positions and pivot columns of the matrix A.

2.4 Solutions of Linear Systems

Recall: Row equivalent matrices correspond to equivalent systems.

Suppose the matrix on the left is the augmented matrix for a linear system of equations in the variables x_1, x_2, x_3, x_4, and x_5. Use the rref to characterize the solution set to the linear system.

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Basic & Free Variables

Definition: Let A be an $m \times n$ matrix that is the coefficient matrix for a system of linear equations in the n variables, $x_1, x_2, \ldots, x_n$. For each $i = 1, \ldots, n$

- if the i^{th} column of A is a pivot column, then x_i is a **basic variable**, and
- if the i^{th} column of A is not a pivot column, the x_i is a **free variable.**

Basic & Free Variables

Consider the system of equations along with its augmented matrix.

$$\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 8x_5 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = & 15 \end{array}$$

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

This is row equivalent to the following rref.

$$\left[\begin{array}{rrrrr|r} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Hence the **basic** variables are x_1, x_2, and x_5, and the **free** variables are x_3 and x_4.

Expressing Solutions

To avoid confusion, i.e., in the interest of clarity, we will **always** write solution sets by expressing basic variables in terms of free variables. We will not write free variables in terms of basic. That is, the solution set to the system whose augmented matrix is row equivalent to

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

will be written

$$\begin{array}{rcl} x_1 & = & 3+2x_3 \\ x_2 & = & 2+2x_3 \\ x_3 & & \text{is free} \\ x_4 & = & 0 \end{array}, \quad \text{i.e.,} \quad \begin{array}{rcl} x_1 & = & 3+2t \\ x_2 & = & 2+2t \\ x_3 & = & t \\ x_4 & = & 0 \end{array}, \quad t \in R.$$

Proper Solution Set Expressions

We will never express free variables in terms of basic variables. All three of the following result from the same augmented matrix:

$$\begin{aligned} x_1 &= 3+2t \\ x_2 &= 2+2t \\ x_3 &= t \\ x_4 &= 0 \end{aligned} \qquad \begin{aligned} x_3 &= -3/2+1/2x_1 \\ x_2 &= 2+2t \\ x_3 &= t \\ x_4 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= 3+2t \\ x_2 &= -1+x_1 \\ x_3 &= t \\ x_4 &= 0 \end{aligned}$$

The left most parametric description is correct. The two expressions in red are **not correct** descriptions. They both include convoluted descriptions of the relationships between the variables.

Main Existence & Uniqueness Theorem

Let A and $\widehat{A}$ be the coefficient matrix and the augmented matrix, respectively of a system of linear equations.

1. If the rightmost column of $\widehat{A}$ is a pivot column of $\widehat{A}$, then the system is inconsistent.
2. If the rightmost column of $\widehat{A}$ is not a pivot column of $\widehat{A}$, then the system is consistent.

Moreover, if the system is consistent, then

1. if every column of A is a pivot column of A, then the system has a unique solution; and
2. if at least one column of A is not a pivot column of A, then the system has infinitely many solutions.

Immediate Corollary

If a system has m equations in n variables, with $m < n$, then the system is either inconsistent or has infinitely many solutions.

$$m \left\{ \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{n>m} \right. ,$$