# September 4 Math 2306 sec. 51 Fall 2024

## Section 4: First Order Equations: Linear

Recall that a first order linear equation is one that has the form[1]

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

In **standard form**, a first order linear equation looks like

$$\frac{dy}{dx} + P(x)y = f(x).$$

Assuming $P$ and $f$ are continuous on some interval $I$, the **general solution** will be the sum of a complementary and a particular solution.

$$y(x) = y_c(x) + y_p(x).$$

[1] It's called homogeneous if $g(x) = 0$ and nonhomogeneous otherwise.

## Solution Process $1^{st}$ Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor $\mu$. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for $y$.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx$$

$$y(x) = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

# Example

$\frac{du}{dx} + P(x)u = f(x)$

Find the general solution of the differential equation

$$\frac{du}{dx} - \frac{4}{x}u = -2x^3.$$

It's in standard form w/ $P(x) = \frac{-4}{x}$

$$\mu = e^{\int P(x)\,dx} = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln|x|}$$

$$= e^{\ln x^{-4}} = x^{-4}$$

multiply by $\mu$

$$x^{-4}\left(\frac{du}{dx} - \frac{4}{x}u\right) = x^{-4}\left(-2x^3\right)$$

$$\frac{d}{dx}\left(x^{-4}u\right) = -2\,x^{-1}$$

$$\int \frac{d}{dx}\left(x^{-4}u\right)dx = \int -2\,x^{-1}\,dx$$

$$x^{-4}u = -2\ln|x| + C$$

Divide by $x^{-4}$

$$u = \frac{-\ln x^2 + C}{x^{-4}}$$

The general solution is

.

$$\boxed{u = x^4 \left( c - \ln x^2 \right)}$$

# Bernoulli Equations

**Bernoulli $1^{st}$ Order Equation**

Suppose $P(x)$ and $f(x)$ are continuous on some interval $(a, b)$ and $n$ is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

**Observation:** A Bernoulli equation looks like a linear one at first glance. However, since $n \neq 0, 1$ a Bernoulli equation is necessarily **nonlinear**.

# Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{1}$$

We'll solve (1) by using a change of variables

$$u = y^{1-n}.$$

The new variable $u$ will satisfy a linear equation which we will solve and substitute back $y = u^{\frac{1}{1-n}}$.

We'll replace the terms in the ODE with expressions in $u$.

$$u = y^{1-n}, \qquad \frac{du}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx}$$

$$= (1-n)y^{-n}\frac{dy}{dx}$$

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Isolate $\frac{dy}{dx}$ $\qquad \frac{1}{1-n} y^n \frac{du}{dx} = \frac{dy}{dx}$

Replace $\frac{dy}{dx}$ in the ODE

$$\frac{1}{1-n} y^n \frac{du}{dx} + P(x)y = f(x)y^n$$

multiply by $\frac{1-n}{y^n}$

$$\frac{du}{dx} + (1-n)P(x)\frac{y}{y^n} = (1-n)f(x)\cancel{\frac{y^n}{y^n}}$$

$$\frac{du}{dx} + (1-n)P(x)\underbrace{y^{1-n}}_{u} = (1-n)f(x)$$

So $u$ solves the linear ODE

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x).$$

We could write this as

$$\frac{du}{dx} + P_1(x)u = f_1(x)$$

where $P_1(x) = (1-n)P(x)$ and $f_1(x) = (1-n)f(x)$

## Solving a Bernoulli Equation $\frac{dy}{dx} + P(x)y = f(x)y^n$

- Introduce the new dependent variable $u = y^{1-n}$.
- Then $u$ solves the first order linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x).$$

- Solve this linear equation using an integrating factor (in the usual way).
- Substitute back to the original variable

$$y = u^{\frac{1}{1-n}}.$$

# Example

Solve the initial value problem

$$\frac{dy}{dx} + \frac{2}{x}y = x^3y^3, \quad x > 0, \quad y(1) = \frac{1}{2}.$$

It's Bernoulli. w/ $n=3$.

$P(x) = \frac{2}{x}$, $f(x) = x^3$, $n=3$

$1-n = 1-3 = -2$

$u = y^{1-n} = y^{1-3} = y^{-2} \Longrightarrow y = u^{-1/2}$

$u$ solves $\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x)$

$$\frac{du}{dx} + (-2)\frac{2}{x}u = (-2)x^3$$

$$\frac{du}{dx} - \frac{4}{x}u = -2x^3$$

we found $u = x^4(c - \ln x^2)$

$u = y^{1-3} = \frac{1}{y^2}$ $\quad y = \frac{1}{\sqrt{u}}$ or $y = \frac{-1}{\sqrt{u}}$

Since $y(1) = \frac{1}{2}$, $y = \frac{+1}{\sqrt{u}}$

y can't be negative since y(1) is positive

$$y = \frac{1}{\sqrt{x^4(c - \ln x^2)}}$$

Apply the I.C.

$$y(1) = \frac{1}{\sqrt{1^4(c - \ln 1^2)}} = \frac{1}{2}$$

$$\frac{1}{\sqrt{c}} = \frac{1}{2} \Rightarrow c = 4$$

The solution to the IVP is

$$\boxed{y = \frac{1}{\sqrt{x^4(4 - \ln x^2)}}}$$

# Example: Logistic Equation

Assume that $M$ and $k$ are positive constants. Show that the following equation is a Bernoulli equation and find the general solution.

$$\frac{dP}{dt} = kP(M-P)$$