September 4 Math 2306 sec. 53 Fall 2024

Section 4: First Order Equations: Linear

Recall that a first order linear equation is one that has the form[1]

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

In **standard form**, a first order linear equation looks like

$$\frac{dy}{dx} + P(x)y = f(x).$$

Assuming P and f are continuous on some interval I, the **general solution** will be the sum of a complementary and a particular solution.

$$y(x) = y_c(x) + y_p(x).$$

[1] It's called homogeneous if $g(x) = 0$ and nonhomogeneous otherwise.

Solution Process 1^{st} Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx$$

$$y(x) = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Example

Find the general solution of the differential equation

$$\frac{du}{dx} - \frac{4}{x}u = -2x^3.$$

It's linear in standard form w/ $P(x) = \frac{-4}{x}$

$$\mu = e^{\int P(x)dx} = e^{\int \frac{-4}{x}dx} = e^{-4 \ln|x|}$$

$$= e^{\ln x^{-4}} = x^{-4}$$

Multiply by μ and collapse

$$x^{-4}\left(\frac{du}{dx} - \frac{4}{x}u\right) = x^{-4}\left(-2x^3\right)$$

$$\frac{d}{dx}\left(x^{-4}u\right) = -2x^{-1}$$

$$\int \frac{d}{dx}\left(x^{-4}u\right)dx = \int -2x^{-1}\,dx$$

$$x^{-4}u = -2\ln|x| + C$$

$$u = \frac{-\ln x^2 + C}{x^{-4}}$$

The solution

$$\boxed{u = x^4\left(C - \ln x^2\right).}$$

Bernoulli Equations

Bernoulli 1^{st} Order Equation

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: A Bernoulli equation looks like a linear one at first glance. However, since $n \neq 0, 1$ a Bernoulli equation is necessarily **nonlinear**.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{1}$$

We'll solve (1) by using a change of variables

$$u = y^{1-n}.$$

The new variable u will satisfy a linear equation which we will solve and substitute back $y = u^{\frac{1}{1-n}}$.

We'll replace the terms in the ODE with expressions in terms of u.

Set $u = y^{1-n}$, $\quad \frac{du}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx}$

$$= (1-n)y^{-n}\frac{dy}{dx}$$

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Multiply by $\frac{1}{1-n} y^n$

$$\frac{dy}{dx} = \frac{1}{1-n} y^n \frac{du}{dx}$$

Put this into the ODE

$$\frac{1}{1-n} y^n \frac{du}{dx} + P(x) y = f(x) y^n$$

Multiply by $\frac{(1-n)}{y^n}$

$$\frac{du}{dx} + (1-n) P(x) \frac{y}{y^n} = (1-n) \cdot f(x) \frac{y^n}{y^n}$$

$$\frac{du}{dx} + (1-n)P(x)\underbrace{y^{1-n}}_{u} = (1-n)f(x)$$

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x)$$

This is linear.

Solving a Bernoulli Equation $\frac{dy}{dx} + P(x)y = f(x)y^n$

- Introduce the new dependent variable $u = y^{1-n}$.
- Then u solves the first order linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x).$$

- Solve this linear equation using an integrating factor (in the usual way).
- Substitute back to the original variable

$$y = u^{\frac{1}{1-n}}.$$

Example

Solve the initial value problem

$$\frac{dy}{dx} + \frac{2}{x}y = x^3y^3, \quad x > 0, \quad y(1) = \frac{1}{2}.$$

It's Bernoulli w/ $n = 3$.

$P(x) = \frac{2}{x}$, $f(x) = x^3$, $n = 3$, $1-n = 1-3 = -2$

Set $u = y^{1-n} = y^{1-3} = y^{-2}$

u solves

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x)$$

$$\frac{du}{dx} + (-2)\frac{2}{x}u = (-2)x^3$$

$$\frac{du}{dx} - \frac{4}{x}u = -2x^3$$

we solved this and found

$$u = x^4(C - \ln x^2)$$

$$u = y^{-2} = \frac{1}{y^2} \Rightarrow y = \frac{1}{\sqrt{u}} \text{ or } y = \frac{-1}{\sqrt{u}}$$

Given $y(1) = \frac{1}{2}$ so y is positive.

so $y = \frac{1}{\sqrt{u}}$

$$y = \frac{1}{\sqrt{x^4(c - \ln x^2)}}$$

Apply $y(1) = \frac{1}{2}$

$$y(1) = \frac{1}{\sqrt{1^4(c - \ln 1^2)}} = \frac{1}{2}$$

$$\frac{1}{\sqrt{c}} = \frac{1}{2} \Rightarrow c = 4$$

The solution to the IVP is

$$y = \frac{1}{\sqrt{x^4(4 - \ln x^2)}}$$