September 6 Math 2306 sec. 51 Fall 2024

Section 4: First Order Equations: Linear

We have solution techniques for first order linear,

$$\frac{dy}{dx} + P(x)y = f(x),$$

and first order Bernoulli equations

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \quad n \neq 0, 1.$$

Homework Notice

In the workbook, linear equations are exercises 1–7 in section 4, and Bernoulli equations are exercises 13 and 14, also in section 4. (Exercises 8–12 can be ignored.)

Solution Process 1^{st} Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx$$

$$y(x) = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Solving a Bernoulli Equation $\frac{dy}{dx} + P(x)y = f(x)y^n$

- Introduce the new dependent variable $u = y^{1-n}$.
- Then u solves the first order linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x).$$

- Solve this linear equation using an integrating factor (in the usual way).
- Substitute back to the original variable

$$y = u^{\frac{1}{1-n}}.$$

Example: Logistic Equation

Assume that M and k are positive constants. Show that the following equation is a Bernoulli equation and find the general solution.

$$\frac{dP}{dt} = kP(M-P) = kMP - kP^2$$

$$\frac{dP}{dt} + Q(t)P = f(t)P^n$$

$$\Rightarrow \frac{dP}{dt} - kMP = -kP^2$$

Bernoulli w/ $n=2$, $Q(t) = -kM$, $f(t) = -k$

Let $u = P^{1-n} = P^{1-2} = P^{-1} \iff u = \frac{1}{P}$ and $P = \frac{1}{u}$

u solves the linear eqn

$$\frac{du}{dt} + (1-n)Q(t)u = (1-n)f(t)$$

$$\frac{du}{dt} + (-1)(-kM)u = (-1)(-k)$$

$$\frac{du}{dt} + kMu = k$$

$$Q(t) = kM, \quad \mu = e^{\int Q(t)\,dt} = e^{\int kM\,dt}$$

$$= e^{kMt}$$

$$\frac{d}{dt}\left(e^{kMt} u\right) = k e^{kMt}$$

$$\int \frac{d}{dt}\left(e^{kMt} u\right) dt = \int k e^{kMt}\,dt$$

$$e^{kMt} u = \cancel{k}\left(\frac{1}{\cancel{k}M}\right) e^{kMt} + C$$

$$u = \frac{\frac{1}{M} e^{kMt} + C}{e^{kMt}}$$

$$u = \frac{1}{M} + C e^{-kMt}$$

$$u = P^{-1} \implies P = \frac{1}{u}$$

$$u = \frac{1}{M} + \frac{CM}{M} e^{-kMt}$$

$$= \frac{1 + A e^{-kMt}}{M} \quad , \quad A = CM$$

Finally, $P(t) = \dfrac{M}{1 + A e^{-kMt}}$

Section 5: First Order Equations: Models and Applications

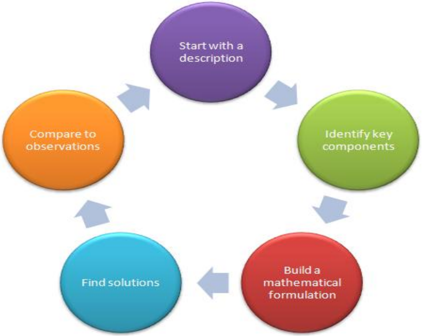

Figure: Mathematical Models give Rise to Differential Equations

In this section, we will consider select models involving first order ODEs. Let's see the process in action.

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2021, there were 58 rabbits. In 2022, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2031.

We need variables. The population is changing in time, so let's introduce variables

$$t \sim \text{time} \quad \text{and} \quad P(t) \sim \text{is the population (density) at time } t.$$

We need to express the following mathematically:

The population's rate of change is proportional to the population.

$$\frac{dP}{dt} = kP$$

rate of change

proportional to P

k is a constant of proportionality.

Let's take t in years with $t=0$ in 2021. With this, we're given

$P(0) = 58$ and $P(1) = 89$

Coupling the ODE w/ $P(0) = 58$,

we have an IVP

$$\frac{dP}{dt} = kP \quad , \quad P(0) = 58$$

Separate variables

$$\frac{1}{P}\frac{dP}{dt} = k$$

$$\int \frac{1}{P}\, dP = \int k\, dt$$

$$\ln P = kt + C$$

$$P = e^{kt+C} = e^{C} e^{kt}$$

Let $A = e^{C}$, $P(t) = Ae^{kt}$

Apply $P(0) = 58$, $P(0) = Ae^{0} = 58$

so $P(t) = 58\, e^{kt}$ is the population @ time t-years.

Apply $P(1) = 89$, we can find k.

$$P(1) = 89 = 58\, e^{k(1)} = 58\, e^{k}$$

$$e^k = \frac{89}{58} \Rightarrow k = \ln\left(\frac{89}{58}\right)$$

The population $P(t) = 58\, e^{t \ln\left(\frac{89}{58}\right)}$

The model predicts a population

$$P(10) = 58\, e^{10 \ln\left(\frac{89}{58}\right)} \approx 4200$$

in 2031.

We could write

$$P(t) = \frac{89^t}{58^{t-1}}$$

Exponential Growth or Decay

Exponential Growth/Decay

If a quantity P changes continuously at a rate proportional to its current value, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear.

If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

In practice, we typically take $k > 0$ and in the case of decay, we write $\frac{dP}{dt} = -kP$.

Series Circuits: RC-circuit

With the restriction that we are considering only models involving first order equations, we can consider two types of simple circuits, an *RC*-series circuit or an *LR*-series circuit.

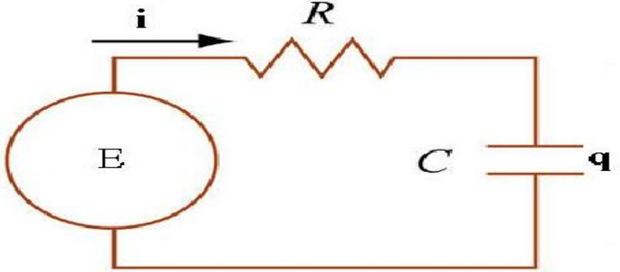

Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge on the capacitor is q and the current $i = \frac{dq}{dt}$. Both q and i are functions of time.

Series Circuits: LR-circuit

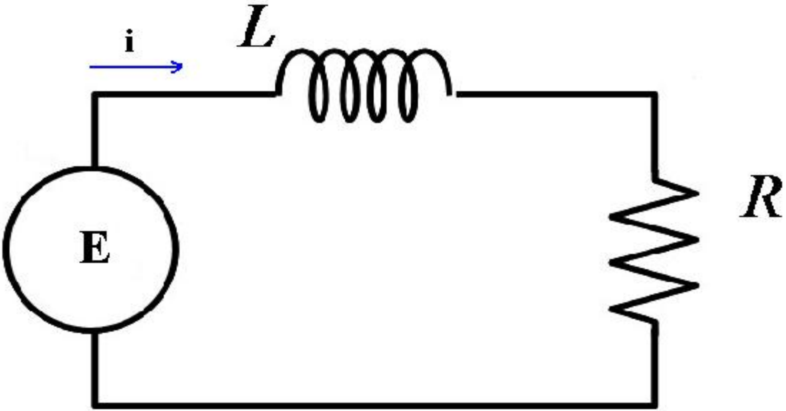

Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. We track the current i as a function of time.

Measurable Quantities:

In these problems, there are several measurable quantities. These are listed here along with the relevant units of measure.

Resistance R in ohms (Ω),	Implied voltage E in volts (V),
Inductance L in henries (h),	Charge q in coulombs (C),
Capacitance C in farads (f),	Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Table: The potential drop across various elements is know empirically.

Kirchhoff's Law

Kirchhoff's Law

Kirchhoff's Law states that:
The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force. We can use this to arrive at a differential equation for the charge $q(t)$ in an RC circuit or the current $i(t)$ in and LR circuit.

Both of these result in a first order linear differential equation.

RC Series Circuit

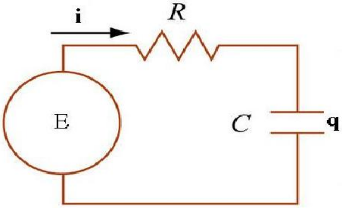

Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

$$\begin{array}{ccccc} \text{drop across resistor} & + & \text{drop across capacitor} & = & \text{applied force} \\ R\frac{dq}{dt} & + & \frac{1}{C}q & = & E(t) \end{array}$$

$$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t)}$$

If $q(0) = q_0$, the IVP can be solved to find $q(t)$ for all $t > 0$.

LR Series Circuit

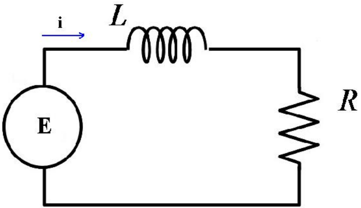

Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

$$\begin{array}{ccccc} \text{drop across inductor} & + & \text{drop across resistor} & = & \text{applied force} \\ L\frac{di}{dt} & + & Ri & = & E(t) \end{array}$$

$$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

If $i(0) = i_0$, the IVP can be solved to find $i(t)$ for all $t > 0$.

Summary of First Order Circuit Models

Before considering an example, let's summarize our two circuit models.

The charge $q(t)$ at time t on the capacitor in an RC-series circuit with resistance R ohm, capacitance C farads, and applied voltage $E(t)$ volts satisfies

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t), \quad q(0) = q_0$$

where q_0 is the initial charge on the capacitor.

The current $i(t)$ at time t in an LR-series circuit with resistance R ohm, inductance L henries, and applied voltage $E(t)$ volts satisfies

$$L\frac{di}{dt} + Ri = E(t), \quad i(0) = i_0$$

where i_0 is the initial current in the circuit.