September 1 Math 2306 sec. 51 Spring 2023

We have three types of first order equations that we know how to solve. These are

$$\text{Separable:} \quad \frac{dy}{dx} = g(x)h(y)$$

$$\text{Linear:} \quad \frac{dy}{dx} + P(x)y = f(x)$$

$$\text{Bernoulli:} \quad \frac{dy}{dx} + P(x)y = f(x)y^n, \quad n \neq 0, 1$$

Remark: It's possible that an equation can be of more than one type. Possible combinations are:

- separable and linear, and
- separable and Bernoulli.

Solving Separable Equations

- Identify the equation as separable $\frac{dy}{dx} = g(x)h(y)$.
- Identify any constant solutions, $y(x) = c$, if possible[a] by solving $h(c) = 0$.
- Separate variables (using multiplication and division)

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

- Integrate both sides of the fully separated equation.

[a] If h is very complicated, it might be necessary to skip this.

Solution Process 1^{st} Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx$$

$$y(x) = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Solving a Bernoulli Equation $\frac{dy}{dx} + P(x)y = f(x)y^n$

- Introduce the new dependent variable $u = y^{1-n}$.
- Then u solves the first order linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x).$$

- Solve this linear equation using an integrating factor (in the usual way).
- Substitute back to the original variable

$$y = u^{\frac{1}{1-n}}.$$

Identifying Equation Types

Classify each equation as being Bernoulli, Linear, or Separable.

(a) $x^2 \dfrac{dy}{dx} = y\cos(2y)$ $\Longrightarrow$ $y' = \frac{1}{x^2}(y\cos(2y))$

It's separable w/ $g(x) = \frac{1}{x^2}$

$h(y) = y\cos(2y)$

(b) $e^{2x} \dfrac{dy}{dx} = y^2 - 2y$ $\Longrightarrow$ $\frac{dy}{dx} = e^{-2x}(y^2 - 2y)$ Separable

Also $\frac{dy}{dx} + 2e^{-2x}y = e^{-2x}y^2$

Bernoulli

This one is both separable and Bernoulli

Identifying Equation Types

Classify each equation as being Bernoulli, Linear, or Separable.

(c) $\frac{dV}{dt} = -2(V-1)$ Separable $g(t) = -2$, $h(V) = V-1$

also $\frac{dV}{dt} + 2V = 2$

Linear

This is both separable and linear.

(d) $\frac{dx}{dt} + \frac{x}{t+1} = \ln(t)$ $\frac{dx}{dt} + \frac{1}{t+1}x = \ln t$

Linear (only)