September 7 Math 2306 sec. 52 Fall 2022

Section 5: First Order Equations Models and Applications

We have three models in the form of first order ODEs:

> If a quantity *P* changes at a rate proportional to itself, it exhibits exponential
>
> $$\text{growth} \quad \frac{dP}{dt} = kP \quad \text{or decay} \quad \frac{dP}{dt} = -kP$$
>
> where *k* is a positive constant.

Examples include population dynamics (over small time intervals), continuously compounding interest, and radio active decay.

RC or LR Circuit

The charge $q(t)$ at time t on the capacitor in an RC-series circuit with resistance R ohm, capacitance C farads, and applied voltage $E(t)$ volts satisfies

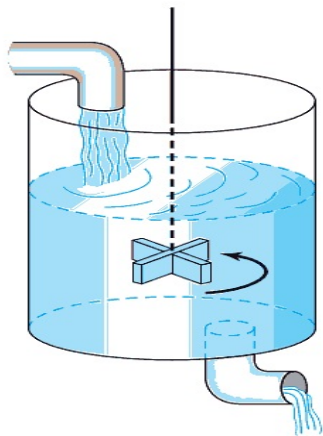$$R\frac{dq}{dt} + \frac{1}{C}q = E(t), \quad q(0) = q_0$$

where q_0 is the initial charge on the capacitor.

The current $i(t)$ at time t in an LR-series circuit with resistance R ohm, inductance L henries, and applied voltage $E(t)$ volts satisfies

$$L\frac{di}{dt} + Ri = E(t), \quad i(0) = i_0$$

where i_0 is the initial current in the circuit.

A Classic Mixing Problem

A composite fluid is kept *well mixed* (i.e. spatially homogeneous).

Figure: We wish to track the amount of some substance in a composite mixture such as salt and water, gas and ethanol, polutant and water, etc. Fluid may flow in and out of the composition, and we assume instant mixing so that the mass of some substance is dependent on time, but not on space.

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.
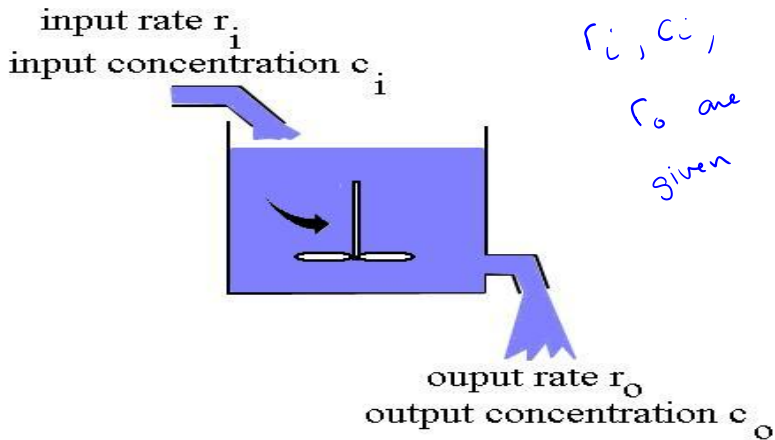
A Classic Mixing Problem

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} input\ rate \\ of\ salt \end{array}\right) - \left(\begin{array}{c} output\ rate \\ of\ salt \end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

$$C_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

initial volume $\nearrow$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i \quad , \quad A(0) = \text{something}$$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$V(0) = 500 \text{ gal}$

$C_i = 2 \text{ lb}/\text{gal}$

$r_i = 5 \text{ gal}/\text{min}$

$r_0 = 5 \text{ gal}/\text{min}$

$V(t) = V(0) + (r_i - r_0)t$

$\qquad = 500 + (5-5)t = 500$

$C_0 = \dfrac{A}{V} = \dfrac{A}{500}$

$\dfrac{dA}{dt} = r_i C_i - r_0 C_0 = 5(2) - 5\left(\dfrac{A}{500}\right)$

$$\Rightarrow \quad \frac{dA}{dt} + \frac{1}{100} A = 10 \quad , \quad A(0) = 0$$

pure water $\Rightarrow$ no salt so $A(0) = 0$

$$\frac{dy}{dt} + P(t) \, y = f(t)$$

$$P(t) = \frac{1}{100} \quad , \quad \mu = e^{\int P(t) dt} = e^{\int \frac{1}{100} dt} = e^{\frac{1}{100} t}$$

$$\frac{d}{dt} \left(e^{\frac{1}{100} t} A \right) = 10 e^{\frac{1}{100} t}$$

$$\int \frac{d}{dt} \left(e^{\frac{1}{100} t} A \right) dt = \int 10 e^{\frac{1}{100} t} dt$$

$$e^{\frac{1}{100}t} A = 10(100) e^{\frac{1}{100}t} + C$$

$$\Rightarrow A = \frac{1000 \, e^{\frac{1}{100}t} + C}{e^{\frac{1}{100}t}} = 1000 + C \, e^{\frac{-1}{100}t}$$

Apply $A(0) = 0$

$$A(0) = 1000 + C \, e^{0} = 0 \Rightarrow C = -1000$$

The amount of salt at time t is

$$A(t) = 1000 - 1000 \, e^{\frac{-1}{100}t}$$

Let C be the concentration of salt in the tank @ t = 5 minutes

$$C = \frac{A(5)}{V(5)} = \frac{1000 - 1000 e^{\frac{-5}{100}} \text{ lb}}{500 \text{ gal.}} \approx 0.01 \quad \frac{\text{lb}}{\text{gal.}}$$

What is C as $t \to \infty$?

$$\lim_{t \to \infty} \frac{A(t)}{V(t)} = \lim_{t \to \infty} \frac{1000 - 1000 e^{-\frac{1}{100}t}}{500} = \frac{1000}{500}$$

$$= 2$$