September 8 Math 2306 sec. 51 Fall 2021

Section 5: First Order Equations Models and Applications

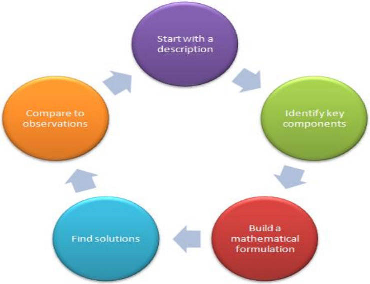

Figure: We've seen exponential growth/decay and simple linear circuits (RC or LR)

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**
Solve this equation and show that for any initial population $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

The ODE is a Bernoulli eqn.

$$\frac{dP}{dt} - kMP = -kP^2 \quad , \quad n = 2$$

$$\frac{dy}{dx} + Q(x)y = f(x)y^n \qquad u = y^{1-n}$$

u solves $\quad \frac{du}{dx} + (1-n)Q(x)u = (1-n)f(x)$

Let $u = P^{1-2} = P^{-1}$

$Q(t) = -kM$ $\quad 1-n = -1$

$f(t) = -k$

so $\frac{du}{dt} + (-1)(-kM)u = (-1)(-k)$

$$\frac{du}{dt} + kMu = k$$

$$Q_1(t) = kM, \quad \mu = e^{\int Q(t)dt} = e^{\int kM\,dt}$$

$$= e^{kMt}$$

$$e^{kMt}(u' + kMu) = k e^{kMt}$$

$$\frac{d}{dt}\left[e^{kMt}u\right] = k e^{kMt}$$

$$\int \frac{d}{dt}\left[e^{kMt}u\right] dt = \int k e^{kMt} dt$$

$$e^{kMt}u = \frac{1}{M} e^{kMt} + C$$

$$u = \frac{1}{M} + C e^{-kMt}$$

$$u = P^{-1} \Rightarrow P = u^{-1}$$

$$P = \frac{1}{\frac{1}{M} + C e^{-kMt}} \cdot \frac{M}{M} = \frac{M}{1 + CM e^{-kMt}}$$

Let $P(0) = P_0$

$$P(0) = \frac{M}{1 + CMe^{0}} = P_0$$

$$M = P_0(1 + CM) = P_0 + P_0 M C$$

$$P_0 M C = M - P_0$$

$$C = \frac{M - P_0}{M P_0}$$

$$P(t) = \frac{M}{1 + \left(\frac{M - P_0}{\cancel{M} P_0}\right)\cancel{M} e^{-kMt}} \quad \frac{P_0}{P_0}$$

$$\boxed{P(t) = \frac{M P_0}{P_0 + (M - P_0) e^{-kMt}}}$$

We can look at the long term population taking $t \to \infty$.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{M P_0}{P_0 + (M - P_0)\, e^{-kMt}}$$

$$= \frac{M P_0}{P_0 + 0} = \frac{M P_0}{P_0} = M$$

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition.**

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \dots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I. \tag{1}$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

NOTE: Taking all of the c's to be zero will **always** satisfy equation (1). The set of functions is linearly **independent** if taking all of the c's equal to zero is the **only** way to make the equation true.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

We want to show that $c_1 f_1(x) + c_2 f_2(x) = 0$ for all x is only true if $c_1 = c_2 = 0$.

Suppose $c_1 \sin x + c_2 \cos x = 0$ for all real x.

The equation has to hold when $x=0$.

We get $c_1 \underbrace{\sin(0)}_{0} + c_2 \underbrace{\cos(0)}_{1} = 0$

This equation is $c_1 (0) + c_2 (1) = 0 \Rightarrow c_2 = 0$.

Now the equation becomes

$$c_1 \sin x = 0 \quad \text{for all } x.$$

This holds when $x = \frac{\pi}{2}$. That is

$$c_1 \sin \frac{\pi}{2} = 0$$

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Hence $c_1 = 0$.

Thus $c_1 f_1(x) + c_2 f_2(x) = 0$ for all x only if $c_1 = c_2 = 0$.

We conclude that $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are <u>linearly independent</u>.