September 8 Math 2306 sec. 54 Fall 2021

Section 5: First Order Equations Models and Applications

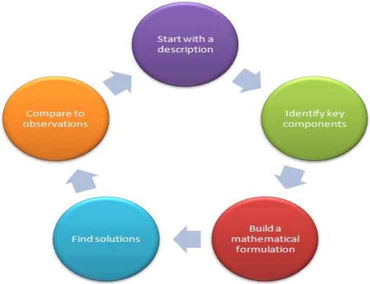

Figure: We've seen exponential growth/decay and simple linear circuits (RC or LR)

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M - P), \quad k, M > 0$$

is called a **logistic growth equation.**
Solve this equation and show that for any initial population $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

Bernoulli $\quad \frac{dy}{dx} + Q(x)y = f(x)y^n$

set $u = y^{1-n}$ then u solves

$$\frac{du}{dx} + (1-n)Q(x)u = (1-n)f(x)$$

Our ODE is $\quad \frac{dP}{dt} - kMP = -kP^2 \quad n = 2$

Set $u = P^{1-2} = P^{-1}$

Here $Q(t) = -kM$, $f(t) = -k$

$1-n = 1-2 = -1$

Hence $$\frac{du}{dt} + (-1)(-kM)u = (-1)(-k)$$

$$\frac{du}{dt} + kMu = k$$

$Q_1(t) = kM$, $\mu = e^{\int Q_1(t)dt} = e^{\int kM dt} = e^{kMt}$

$$e^{kMt}\left(\frac{du}{dt} + kMu\right) = ke^{kMt}$$

$$\frac{d}{dt}\left[e^{kMt}u\right] = ke^{kMt}$$

$$\int \frac{d}{dt}\left[e^{kMt}u\right]dt = \int k e^{kMt}\,dt$$

$$e^{kMt}u = \frac{1}{M}e^{kMt} + C$$

$$u = \frac{1}{M} + Ce^{-kMt}$$

$$u = P^{-1} \implies P = u^{-1} = \frac{1}{u}$$

So $$P(t) = \frac{1}{\frac{1}{M} + Ce^{-kMt}} \cdot \frac{M}{M}$$

$$P(t) = \frac{M}{1 + CMe^{-kMt}}$$

Let $P(0) = P_0$, and let's find C

$$P(0) = \frac{M}{1 + CMe^0} = P_0$$

$$\Rightarrow \quad M = P_0(1 + CM) = P_0 + P_0 MC$$

$$M - P_0 = P_0 MC$$

$$C = \frac{M - P_0}{MP_0}$$

$$P(t) = \frac{M}{1 + CMe^{-kMt}}$$

$$P = \frac{M}{1 + \left(\frac{M-P_0}{\cancel{M}P_0}\right)\cancel{M}e^{-kMt}} \cdot \frac{P_0}{P_0}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}}$$

This is the solution to the IVP.

Looking at the long term solution

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}$$

$$= \frac{MP_0}{P_0 + (M-P_0)\cdot 0} = \frac{MP_0}{P_0} = M$$

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

to solve subject to conditions

$$y(x_0)=y_0,\quad y'(x_0)=y_1,\quad \ldots,\quad y^{(n-1)}(x_0)=y_{n-1}.$$

The problem is called **homogeneous** if $g(x)\equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition.**

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \dots, c_n$ with at least one of them being nonzero such that

$$\boxed{c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.} \tag{1}$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

NOTE: Taking all of the c's to be zero will **always** satisfy equation (1). The set of functions is linearly **independent** if taking all of the c's equal to zero is the **only** way to make the equation true.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

We need to show that $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real x is only true if $c_1 = 0$ and $c_2 = 0$. The equation is

$$c_1 \sin x + c_2 \cos x = 0.$$

We'll assume this is true for all real x. The equation must be true if $x=0$. When $x=0$, we have

$$c_1 \underbrace{\sin 0}_{0} + c_2 \underbrace{\cos 0}_{1} = 0$$

That is, $c_1(0) + c_2(1) = 0 \Rightarrow c_2 = 0$.

The equation is $c_1 \sin x = 0$. for all real x,

The equation holds when $x = \frac{\pi}{2}$. This gives

$$c_1 \underbrace{\sin\frac{\pi}{2}}_{1} = 0$$

That is, $c_1(1) = 0 \Rightarrow c_1 = 0$.

So $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real x is only true if $c_1 = 0$ and $c_2 = 0$.

It follows that $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are <u>linearly independent</u>.