

Chapter 2 Systems of Linear Equations

In this chapter, we will consider equations that have a special structure known as **linearity**. We will

- ▶ define linear equations and linear systems,
- ▶ define solutions and solution sets,
- ▶ learn Gaussian elimination,
- ▶ introduce matrices as a tool for solving linear systems,
- ▶ and learn how matrices can be used to solve linear systems.

Linear Equation (in n variables)

A **linear equation** in the variables x_1, x_2, \dots, x_n is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where a_1, a_2, \dots, a_n and b are real numbers (scalars). The numbers a_1, \dots, a_n are called the **coefficients**, and b can be called the **constant term**.

A **linear** equation is one in which exactly two operations can be done on the variables: (1) multiply by scalars, and (2) add.

Linear: $2x_1 - \sqrt{2}x_2 + 21x_3 = -1, \quad -3x_1 + 4x_2 = 0$

Not Linear: $x_1x_2 + x_3 = 4, \quad x_1^3 - e^{x_2} = 0$

System of Linear Equations

A **system of linear equations** (a.k.a. a *linear system*) is a collection of one or more linear equations in the same variables considered together. A generic system of m equations in n variables is shown in equation (1).

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \quad (3)$$

For example

$$\underbrace{\begin{array}{ccccccccc} 3x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 5 \\ 2x_1 & & & - & 4x_3 & + & 6x_4 & = & 4 \end{array}}_{\text{2 equations in 4 variables}} \quad \text{or} \quad \underbrace{\begin{array}{ccccccccc} 2x_1 & - & x_2 & - & x_3 & = & 0 \\ 3x_1 & & & - & 2x_3 & = & 0 \\ -x_1 & - & x_2 & + & x_3 & = & 0 \end{array}}_{\text{3 equations in 3 variables}}$$

Homogeneous -vs- Nonhomogeneous

A system is called **homogeneous** if all constant terms are zero. Otherwise, it's called **nonhomogeneous**.

$$\begin{array}{rcccccc} 2x_1 & - & x_2 & - & x_3 & = & 0 \\ 3x_1 & + & 0x_2 & - & 2x_3 & = & 0 \\ -x_1 & - & x_2 & + & x_3 & = & 0 \end{array}$$

a homogeneous system

$$\begin{array}{rcccccc} 3x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 5 \\ 2x_1 & & & - & 4x_3 & + & 6x_4 & = & 4 \end{array}$$

a nonhomogeneous system

Solutions & Solution Sets

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \quad (1)$$

Solutions & Solution Sets

A **solution** of (1) is as an ordered n -tuple of real numbers, (s_1, s_2, \dots, s_n) , having the property that upon substitution,

$$x_1 = s_1, \quad x_2 = s_2, \quad \cdots, \quad x_n = s_n,$$

every equation in the system reduces to an identity. The collection of all solutions of (1) is called the **solution set** of the system.

Example: Show that $(2, 1, 3)$ is a solution of the homogeneous system

$$\begin{array}{rrcr} 2x_1 & - & x_2 & - & x_3 & = & 0 \\ 3x_1 & & & - & 2x_3 & = & 0 \\ -x_1 & - & x_2 & + & x_3 & = & 0 \end{array}$$

We can sub

$$x_1 = 2, x_2 = 1, x_3 = 3$$

into each equation.

1st equation $2(2) - 1 - 3 = 4 - 1 - 3 = 4 - 4 = 0$

2nd equation $3(2) - 2(3) = 6 - 6 = 0$

3rd equation: $-2 - 1 + 3 = -3 + 3 = 0$

All three give $0 = 0$ so $(2, 1, 3)$
is a solution.

The solution set¹ of this system is $\{(2t, t, 3t) \mid t \in R\}$.

$$\begin{array}{rrcr} 2x_1 & - & x_2 & - & x_3 & = & 0 \\ 3x_1 & & & - & 2x_3 & = & 0 \\ -x_1 & - & x_2 & + & x_3 & = & 0 \end{array}$$

we can verify as

before, setting

$$x_1 = 2t, \quad x_2 = t, \quad x_3 = 3t$$

1st equation $2(2t) - (t) - (3t) = 4t - t - 3t = 0$

To complete this verification, we should substitute into the second and third equation and show that all three reduce to $0 = 0$.

¹It's not obvious that there are no solutions other than the ones listed here.

Expressing Solutions

We can use the notation from the previous examples:

- ▶ as a point (s_1, s_2, \dots, s_n) —e.g., $(2, 1, 3)$, or
- ▶ using set builder notation, such as $\{(2t, t, 3t) \mid t \in R\}$

The more common types will be **parametric** and **vector parametric**.

Parametric

A **parametric description**—or **parametric form**—is a list. For example,

$$\begin{array}{rcl} x_1 & = & 2 \\ x_2 & = & 1 \\ x_3 & = & 3 \end{array} \quad \text{or} \quad \begin{array}{rcl} x_1 & = & 2t \\ x_2 & = & t \\ x_3 & = & 3t \end{array} \quad -\infty < t < \infty .$$

Expressing Solutions

Vector Parametric

A **vector parametric description**—or **vector parametric form**—is a solution/solutions expressed as a vector/vectors.

$$\vec{x} = \langle 2, 1, 3 \rangle, \quad \text{or} \quad \vec{x} = t\langle 2, 1, 3 \rangle, \quad t \in R.$$

Remark: Note here that we're just taking our variables, x_1, x_2, \dots, x_n , and placing them as entries in a vector

$$\vec{x} = \langle x_1, x_2, \dots, x_n \rangle.$$

If we have a parametric description (i.e., we already have a list), then it's a simple task of building vector(s) from that list.

$$\begin{array}{rcccccccl} 3x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 5 \\ 2x_1 & & & & - & 4x_3 & + & 6x_4 & = & 4 \end{array} \quad (2)$$

It can be shown that the solutions of (2) are the 4-tuples (x_1, x_2, x_3, x_4) where

- ▶ x_3 and x_4 can be any real numbers as long as
- ▶ $x_1 = 2 + 2x_3 - 3x_4$, and $x_2 = -1 - 4x_3 + 5x_4$.

To write a **parametric** description, we choose *parameter* names for x_3 and x_4 , and then list the formulas. Letting $x_3 = s$ and $x_4 = t$, with the understanding that $s, t \in R$, we can write

$$\begin{array}{rcl} x_1 & = & 2 + 2s - 3t, \\ x_2 & = & -1 - 4s + 5t, \\ x_3 & = & s, \\ x_4 & = & t, \end{array} \quad , \quad s, t \in R.$$

Converting Parametric to Vector Parametric Form

$$\begin{aligned}x_1 &= 2 + 2s - 3t, \\x_2 &= -1 - 4s + 5t, \\x_3 &= s, \\x_4 &= t,\end{aligned}, \quad s, t \in \mathbb{R}.$$

Let's convert this parametric description to vector parametric. (The process will be familiar.)

$$\vec{x} = \langle x_1, x_2, x_3, x_4 \rangle$$

$$= \langle 2 + 2s - 3t, -1 - 4s + 5t, s, t \rangle$$

$$= \langle 2, -1, 0, 0 \rangle + \langle 2s, -4s, s, 0 \rangle + \langle -3, 5, 0, t \rangle$$

$$= \langle 2, -1, 0, 0 \rangle + s \langle 2, -4, 1, 0 \rangle + t \langle -3, 5, 0, 1 \rangle$$

$$s, t \in \mathbb{R}.$$

Equivalent Systems

Definition: Equivalent Systems

We will say that two systems of linear equations are **equivalent** if they have the same solution set.

Remark: Equivalent systems will necessarily be in the same variables, but they don't necessarily **look** the same. They don't even have to have the same number of equations!

$$\begin{array}{rclclcl} x_1 & + & 2x_2 & - & x_3 & = & 2 \\ 3x_1 & + & x_2 & - & x_3 & = & 2 \\ 5x_1 & + & 5x_2 & - & 3x_3 & = & 6 \\ -x_1 & - & 3x_2 & & & = & -7 \end{array} \quad \text{and} \quad \begin{array}{rclclcl} x_1 & + & 2x_2 & - & x_3 & = & 2 \\ & & x_2 & + & x_3 & = & 5 \\ & & & & x_3 & = & 3 \end{array}$$

These two systems are equivalent, but it's not at all obvious!

Existence & Uniqueness

Existence & Uniqueness

Theorem: For a system of linear equations, exactly one of the following is true:

- i. the solution set is empty (i.e., there is no solution),
- ii. there exists a unique solution, or
- iii. there are infinitely many solutions.

If a system has at least one solution, we call it **consistent**. Otherwise, we call the system **inconsistent**. So

- ▶ Case i. is **inconsistent**, and
- ▶ Cases ii. and iii. are **consistent**.

Homogeneous Systems

Consider a homogeneous system in n variables.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

Question: Without more information about the coefficients, a_{ij} , is there an *obvious* solution?

$x_1 = x_2 = x_3 = \cdots = x_n = 0$ is a solution

In vector parametric $\vec{x} = \vec{0}_n$

The Trivial Solution

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

The solution $x_1 = x_2 = \cdots = x_n = 0$, or in vector parametric form

$$\vec{x} = \vec{0}_n,$$

is called the **trivial solution**.

Every homogeneous system is consistent because every homogeneous system admits the trivial solution.

- ▶ If a homogeneous system has **exactly one solution**, then it **must** be the trivial solution.
- ▶ If a homogeneous system has infinitely many solutions, we call any solution that is not the zero vector **a nontrivial solution**.

2.1.1 Systems of Two Equations with Two Variables

A system of two equations in two variables has the form

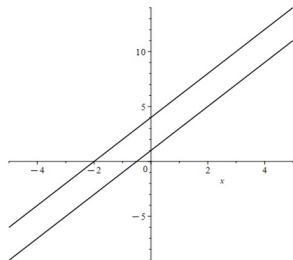
$$\begin{array}{rclcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}.$$

We can think of the two equations as corresponding to a pair of lines, something like

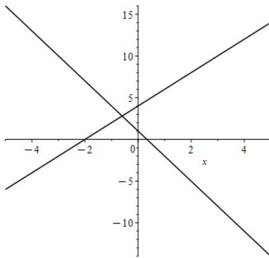
$$x_2 = (\text{slope}) x_1 + (\text{intercept}).$$

Such systems allow us to compare the three solution cases geometrically.

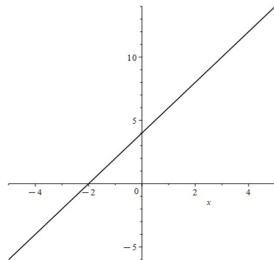
The three solution cases are easily visualized in R^2 .



$$\begin{aligned} 2x_1 - x_2 &= 4 \\ 2x_1 - x_2 &= 1 \end{aligned}$$



$$\begin{aligned} 2x_1 - x_2 &= 4 \\ 3x_1 + x_2 &= -1 \end{aligned}$$



$$\begin{aligned} 2x_1 - x_2 &= 4 \\ -4x_1 + 2x_2 &= -8 \end{aligned}$$

Figure: Lines determined by two linear equations in two variables illustrating the three possible geometric relationships.

- i. parallel, non-intersecting lines correspond to an inconsistent system,
- ii. lines with two distinct slopes correspond to a system with one solution,
- iii. coincident lines corresponds to a system with infinitely many solutions.

Example

Translate each equation into a line and determine if the system is consistent or inconsistent. If consistent, state whether there is a unique solution or infinitely many solutions.

$$\begin{array}{rclcl} 3x_1 & - & x_2 & = & 4 \\ 4x_1 & + & 2x_2 & = & -2 \end{array}$$

$$3x_1 - 4 = x_2 \Rightarrow x_2 = 3x_1 - 4$$

$$2x_2 = -2 - 4x_1 \Rightarrow x_2 = -2x_1 - 1$$

Different slopes one solution.

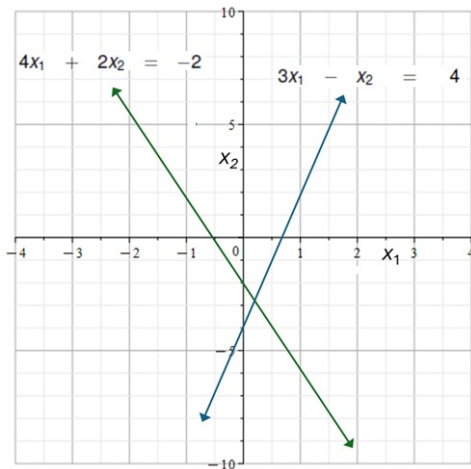


Figure: Lines defined by the equations $3x_1 - x_2 = 4$ and $4x_1 + 2x_2 = -2$ intersect at the point $(\frac{3}{5}, -\frac{11}{5})$. The solution set consists of this single ordered pair.