September 8 Math 3260 sec. 53 Fall 2025

Chapter 2 Systems of Linear Equations

In this chapter, we will consider equations that have a special structure known as **linearity**. We will

- define linear equations and linear systems,
- define solutions and solution sets,
- learn Gaussian elimination,
- introduce matrices as a tool for solving linear systems,
- and learn how matrices can be used to solve linear systems.

Linear Equation (in n variables)

A **linear equation** in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers (scalars). The numbers $a_1, \ldots, a_n$ are called the **coefficients**, and b can be called the **constant term**.

A **linear** equation is one in which exactly two operations can be done on the variables: (1) multiply by scalars, and (2) add.

Linear: $2x_1 - \sqrt{2}\,x_2 + 21x_3 = -1, \quad -3x_1 + 4x_2 = 0$

Not Linear: $x_1x_2 + x_3 = 4, \quad x_1^3 - e^{x_2} = 0$

System of Linear Equations

A **system of linear equations** (a.k.a. a *linear system*) is a collection of one or more linear equations in the same variables considered together. A generic system of m equations in n variables is shown in equation (1).

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}. \qquad (3)$$

For example

$$\underbrace{\begin{array}{ccccccccc} 3x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 5 \\ 2x_1 & & & - & 4x_3 & + & 6x_4 & = & 4 \end{array}}_{\text{2 equations in 4 variables}} \quad \text{or} \quad \underbrace{\begin{array}{ccccccc} 2x_1 & - & x_2 & - & x_3 & = & 0 \\ 3x_1 & + 0x_2 & & - & 2x_3 & = & 0 \\ -x_1 & - & x_2 & + & x_3 & = & 0 \end{array}}_{\text{3 equations in 3 variables}}$$

Homogeneous -vs- Nonhomogeneous

A system is called **homogeneous** if all constant terms are zero. Otherwise, it's called **nonhomogeneous**.

$$\underbrace{\begin{array}{rcrcrcr} 2x_1 & - & x_2 & - & x_3 & = & 0 \\ 3x_1 & & & - & 2x_3 & = & 0 \\ -x_1 & - & x_2 & + & x_3 & = & 0 \end{array}}_{\text{a homogeneous system}}$$

$$\underbrace{\begin{array}{rcrcrcrcr} 3x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 5 \\ 2x_1 & & & - & 4x_3 & + & 6x_4 & = & 4 \end{array}}_{\text{a nonhomogeneous system}}$$

Solutions & Solution Sets

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} . \tag{1}$$

Solutions & Solution Sets

A **solution** of (1) is as an ordered n-tuple of real numbers, $(s_1, s_2, \ldots, s_n)$, having the property that upon substitution,

$$x_1 = s_1, \quad x_2 = s_2, \quad \cdots, \quad x_n = s_n,$$

every equation in the system reduces to an identity. The collection of all solutions of (1) is called the **solution set** of the system.

Example: Show that $(2, 1, 3)$ is a solution of the homogeneous system

$$\begin{array}{rcrcrcl} 2x_1 & - & x_2 & - & x_3 & = & 0 \\ 3x_1 & & & - & 2x_3 & = & 0 \\ -x_1 & - & x_2 & + & x_3 & = & 0 \end{array}.$$

We can sub $x_1 = 2$, $x_2 = 1$, $x_3 = 3$ into all three equations.

1st equation $2(2) - 1 - 3 = 4 - 4 = 0$
1st equation is true

2nd equation $3(2) \quad - 2(3) = 6 - 6 = 0$ also true

3rd equation $-2 - 1 + 3 = -3 + 3 = 0$ also true.

All three equations reduce to the identity $0 = 0$, so $(2, 1, 3)$ is a solution.

The solution set[1] of this system is $\{(2t, t, 3t) \mid t \in R\}$.

$$\begin{array}{rcrcrcl} 2x_1 & - & x_2 & - & x_3 & = & 0 \\ 3x_1 & & & - & 2x_3 & = & 0 \\ -x_1 & - & x_2 & + & x_3 & = & 0 \end{array}.$$

We can verify by setting $x_1 = 2t$, $x_2 = t$, and $x_3 = 3t$ leaving t as t.

1st equation $2(2t) - t - (3t) = 4t - t - 3t = 4t - 4t = 0$

2nd equation $3(2t) \quad -2(3t) = 6t - 6t = 0$

3rd equation $-(2t) - t + 3t = -3t + 3t = 0$

$(2t, t, 3t)$ solves all equations in the system for any value t.

[1] It's not obvious that there are no solutions other than the ones listed here.

Expressing Solutions

We can use the notation from the previous examples:

- as a point $(s_1, s_2, \ldots, s_n)$—e.g., $(2, 1, 3)$, or
- using set builder notation, such as $\{(2t, t, 3t) \mid t \in R\}$

The more common types will be **parametric** and **vector parametric**.

Parametric

A **parametric description**—or **parametric form**—is a list. For example,

$$\begin{array}{rcl} x_1 & = & 2 \\ x_2 & = & 1 \\ x_3 & = & 3 \end{array} \quad \text{or} \quad \begin{array}{rcl} x_1 & = & 2t \\ x_2 & = & t \\ x_3 & = & 3t \end{array} \quad -\infty < t < \infty \ .$$

Expressing Solutions

Vector Parametric

A **vector parametric description**—or **vector parametric form**—is a solution/solutions expressed as a vector/vectors.

$$\vec{x} = \langle 2, 1, 3\rangle, \qquad \text{or} \quad \vec{x} = t\langle 2, 1, 3\rangle, \quad t \in R.$$

"∈" is an element of $t \in R$ "t in R"

Remark: Note here that we're just taking our variables, $x_1, x_2, \dots, x_n$, and placing them as entries in a vector

$$\vec{x} = \langle x_1, x_2, \dots, x_n\rangle.$$

If we have a parametric description (i.e., we already have a list), then it's a simple task of building vector(s) from that list.

$$\begin{array}{rcrcrcrcl} 3x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 5 \\ 2x_1 & & & - & 4x_3 & + & 6x_4 & = & 4 \end{array} \tag{2}$$

It can be shown that the solutions of (2) are the 4-tuples (x_1, x_2, x_3, x_4) where

- x_3 and x_4 can be any real numbers as long as
- $x_1 = 2 + 2x_3 - 3x_4$, and $x_2 = -1 - 4x_3 + 5x_4$.

To write a **parametric** description, we choose *parameter* names for x_3 and x_4, and then list the formulas. Letting $x_3 = s$ and $x_4 = t$, with the understanding that $s, t \in R$, we can write

$$\begin{array}{rcl} x_1 & = & 2 + 2s - 3t, \\ x_2 & = & -1 - 4s + 5t, \\ x_3 & = & s, \\ x_4 & = & t, \end{array} \quad , \quad s, t \in R.$$

$-\infty < s, t < \infty$

Converting Parametric to Vector Parametric Form

$$\begin{array}{rcl} x_1 & = & 2+2s-3t, \\ x_2 & = & -1-4s+5t, \\ x_3 & = & s, \\ x_4 & = & t, \end{array} \quad , \quad s,t \in R.$$

Let's convert this parametric description to vector parametric. (The process will be familiar.)

$\vec{x} = \langle x_1, x_2, x_3, x_4 \rangle$

$= \langle 2+2s-3t, -1-4s+5t, s, t \rangle$

$= \langle 2, -1, 0, 0 \rangle + \langle 2s, -4s, s, 0 \rangle + \langle -3t, 5t, 0, t \rangle$

constant   s stuff   t stuff

$= \langle 2, -1, 0, 0 \rangle + s\langle 2, -4, 1, 0 \rangle + t\langle -3, 5, 0, 1 \rangle$

Equivalent Systems

Definition: Equivalent Systems

We will say that two systems of linear equations are **equivalent** if they have the same solution set.

Remark: Equivalent systems will necessarily be in the same variables, but they don't necessarily **look** the same. They don't even have to have the same number of equations!

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 2 \\ 3x_1 & + & x_2 & - & x_3 & = & 2 \\ 5x_1 & + & 5x_2 & - & 3x_3 & = & 6 \\ -x_1 & - & 3x_2 & & & = & -7 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 2 \\ & & x_2 & + & x_3 & = & 5 \\ & & & & x_3 & = & 3 \end{array}$$

These two systems are equivalent, but it's not at all obvious!

Existence & Uniqueness

Existence & Uniqueness

Theorem: For a system of linear equations, exactly one of the following is true:

i. the solution set is empty (i.e., there is no solution),

ii. there exists a unique solution, or

iii. there are infinitely many solutions.

If a system has at least one solution, we call it **consistent**. Otherwise, we call the system **inconsistent**. So

- Case i. is **inconsistent**, and
- Cases ii. and iii. are **consistent**.

Homogeneous Systems

Consider a homogeneous sytem in n variables.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}.$$

Question: Without more information about the coefficients, a_{ij}, is there an *obvious* solution?

$x_1 = 0,\ x_2 = 0,\ \ldots,\ x_n = 0$ solves the system for any set of coefficients.

The Trivial Solution

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & + & \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}.$$

The solution $x_1 = x_2 = \cdots = x_n = 0$, or in vector parametric form

$$\vec{x} = \vec{0}_n,$$

is called the **trivial solution**.

Every homogeneous system is consistent because every homogeneous system admits the trivial solution.

- If a homogeneous system has **exactly one solution**, then it **must** be the trivial solution.
- If a homogeneous system has infinitely many solutions, we call any solution that is not the zero vector **a nontrivial solution**.

2.1.1 Systems of Two Equations with Two Variables

A system of two equations in two variables has the form

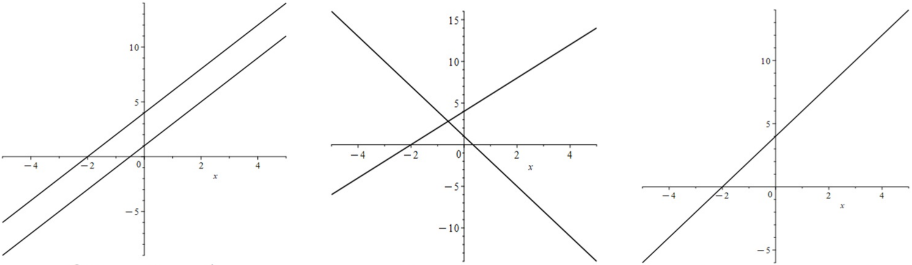
$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}.$$

We can think of the two equations as corresponding to a pair of lines, something like

$$x_2 = (\text{slope})\, x_1 + (\text{intercept}).$$

Such systems allow us to compare the three solution cases geometrically.

The three solution cases are easily visualized in R^2.



$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 4 \\ 2x_1 & - & x_2 & = & 1 \end{array} \qquad \begin{array}{rcrcr} 2x_1 & - & x_2 & = & 4 \\ 3x_1 & + & x_2 & = & -1 \end{array} \qquad \begin{array}{rcrcr} 2x_1 & - & x_2 & = & 4 \\ -4x_1 & + & 2x_2 & = & -8 \end{array}$$

Figure: Lines determined by two linear equations in two variables illustrating the three possible geometric relationships.

i. parallel, non-intersecting lines correspond to an inconsistent system,
ii. lines with two distinct slopes correspond to a system with one solution,
iii. coincident lines corresponds to a system with infinitely many solutions.

Example

Translate each equation into a line and determine if the system is consistent or inconsistent. If consistent, state whether there is a unique solution or infinitely many solutions.

$$\begin{aligned} 3x_1 - x_2 &= 4 \\ 4x_1 + 2x_2 &= -2 \end{aligned}$$

$3x_1 - x_2 = 4 \Rightarrow x_2 = 3x_1 - 4$

$4x_1 + 2x_2 = -2 \Rightarrow 2x_2 = -4x_1 - 2 \Rightarrow x_2 = -2x_1 - 1$

Different slopes $\Rightarrow$ the system is consistent w/ exactly one solution.

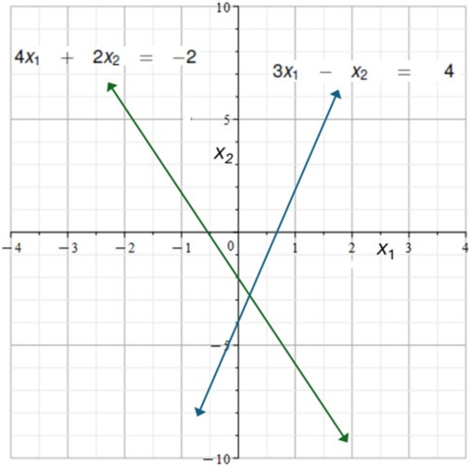


Figure: Lines defined by the equations $3x_1 - x_2 = 4$ and $4x_1 + 2x_2 = -2$ intersect at the point $\left(\frac{3}{5}, -\frac{11}{5}\right)$. The solution set consists of this single ordered pair.

Example

Translate each equation into a line and determine if the system is consistent or inconsistent. If consistent, state whether there is a unique solution or infinitely many solutions.

$$\begin{array}{rcrcr} -5x_1 & + & 3x_2 & = & 6 \\ 2x_1 & - & \frac{6}{5}x_2 & = & -\frac{12}{5} \end{array}$$

$-5x_1 + 3x_2 = 6 \Rightarrow 3x_2 = 5x_1 + 6 \Rightarrow x_2 = \frac{5}{3}x_1 + 2$

$2x_1 - \frac{6}{5}x_2 = \frac{-12}{5} \Rightarrow \frac{6}{5}x_2 = 2x_1 + \frac{12}{5} \Rightarrow x_2 = \frac{5}{6}\left(2x_1 + \frac{12}{5}\right)$

$x_2 = \frac{5}{3}x_1 + 2$

The system is consistent w/ infinitely many solutions.

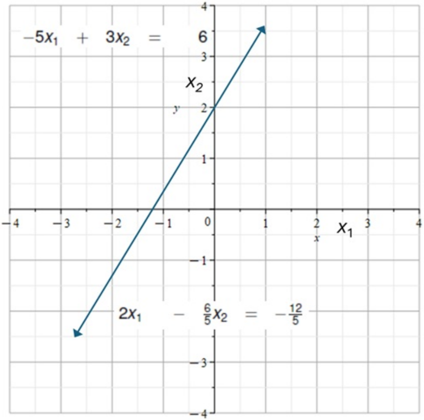


Figure: Lines defined by the equations $-5x_1 + 3x_2 = 6$ and $2x_1 - \frac{6}{5}x_2 = -\frac{12}{5}$ are concurrent. Every pair (x_1, x_2) on this line is in the solution set.

A parametric or vector parametric description could be

$$\begin{array}{lcl} x_1 & = & -\frac{6}{5} + \frac{3}{5}t \\ x_2 & = & t, \quad t \in R \end{array}, \quad \text{or} \quad \vec{x} = \left\langle -\frac{6}{5}, 0 \right\rangle + t \left\langle \frac{3}{5}, 1 \right\rangle$$