September 9 Math 2306 sec. 51 Fall 2024

Section 5: First Order Equations: Models and Applications

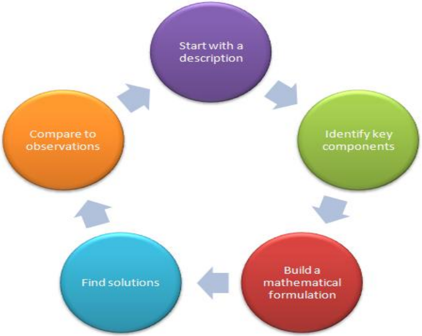


Figure: Mathematical Models give Rise to Differential Equations

In this section, we will consider select models involving first order ODEs. Let's see the process in action.

RC and LR Series Circuits

We want to track the charge on a capacitor (RC circuit) or the current in a circuit (LR circuit). Recall that current i is rate of change of charge q. These are functions of time.

The voltage across each type of element is shown below:

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Table: The potential drop across various elements.

Kirchhoff's Law

Kirchhoff's Law

Kirchhoff's Law states that:
The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force. We can use this to arrive at a differential equation for the charge $q(t)$ in an RC circuit or the current $i(t)$ in and LR circuit.

Both of these result in a first order linear differential equation.

RC Series Circuit

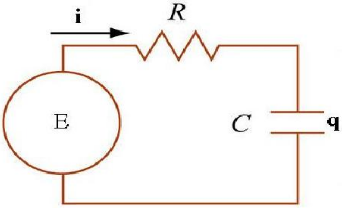


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

drop across resistor	+	drop across capacitor	=	applied force
$R\frac{dq}{dt}$	$+$	$\frac{1}{C}q$	$=$	$E(t)$

$$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t)}$$

If $q(0) = q_0$, the IVP can be solved to find $q(t)$ for all $t > 0$.

LR Series Circuit

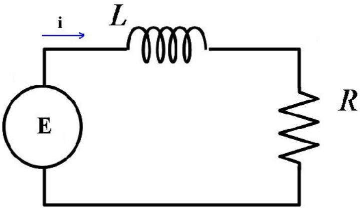


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

drop across inductor	+	drop across resistor	=	applied force
$L\frac{di}{dt}$	$+$	Ri	$=$	$E(t)$

$$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

If $i(0) = i_0$, the IVP can be solved to find $i(t)$ for all $t > 0$.

Summary of First Order Circuit Models

Before considering an example, let's summarize our two circuit models.

The charge $q(t)$ at time t on the capacitor in an RC-series circuit with resistance R ohm, capacitance C farads, and applied voltage $E(t)$ volts satisfies

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t), \quad q(0) = q_0$$

where q_0 is the initial charge on the capacitor.

The current $i(t)$ at time t in an LR-series circuit with resistance R ohm, inductance L henries, and applied voltage $E(t)$ volts satisfies

$$L\frac{di}{dt} + Ri = E(t), \quad i(0) = i_0$$

where i_0 is the initial current in the circuit.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Given $R = 1000\,\Omega$

$C = 5 \cdot 10^{-6}$ f

$E(t) = 200$ V

$$1000\frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}}q = 200, \quad i(0) = q'(0) = 0.4$$

$$\frac{1}{5 \cdot 10^{-6}} = \frac{10^6}{5} = 2 \cdot 10^5$$

Put the ODE in standard form

$$\frac{dq}{dt} + \frac{2\cdot 10^5}{1000} q = \frac{200}{1000}$$

$$\frac{dq}{dt} + 200q = 0.2$$

$$P(t) = 200, \quad \mu = e^{\int 200\,dt} = e^{200t}$$

$$\frac{d}{dt}\left(e^{200t} q\right) = 0.2\,e^{200t}$$

$$\int \frac{d}{dt}\left(e^{200t} q\right) dt = \int 0.2\,e^{200t}\,dt$$

$$e^{200t} q = \frac{0.2}{200} e^{200t} + k$$

$$q = \frac{0.001\,e^{200t} + k}{e^{200t}}$$

$$q = 0.001 + k e^{-200t}$$

Apply the IC $q'(0) = 0.4$

$$q'(t) = 0 - 200 k e^{-200t}$$

$$q'(0) = -200k e^{0} = 0.4$$

$$\Rightarrow k = \frac{0.4}{-200} = -0.002$$

The charge on the capacitor is

$$q(t) = 0.001 - 0.002 e^{-200t}$$

The long time charge

$$\lim_{t \to \infty} q(t) = \lim_{t \to \infty} \left(0.001 - 0.002 e^{-200t}\right) = 0.001 \quad C$$

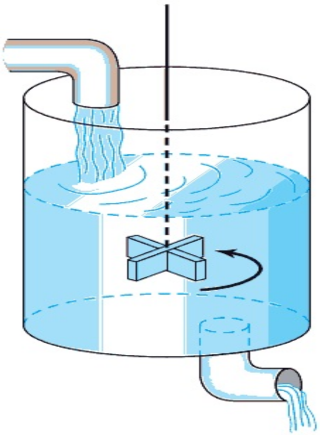
A Classic Mixing Problem

Classical mixing involves tracking the mass of some substance in a composite mixture. Examples include

- salt in a salt-water mixture,
- ethanol in an ethanol-gasoline mixture,
- polutant in a volume of water.

Let's look at a specific problem and build a model that can be used in general. First, a visual.

A Classic Mixing Problem



A composite fluid is kept *well mixed* (i.e. spatially homogeneous).

Figure: We wish to track the amount of some substance in a composite mixture such as salt and water, gas and ethanol, polutant and water, etc. Fluid may flow in and out of the composition, and we assume instant mixing so that the mass of some substance is dependent on time, but not on space.

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

In order to answer such a question, we need to convert the problem statement into a mathematical one.

Some Notation

In addition to the amount of salt, $A(t)$, at time t we have several variables or parameters. Let

- r_i be the rate at which fluid enters the tank (rate in),
- r_o be the rate at which fluid leaves the tank (rate out),
- c_i be the concentration of substance (salt) in the in-flowing fluid (concentration in),
- c_o be the concentration of substance (salt) in the out-flowing fluid (concentration out),
- $V(t)$ be the total volume of fluid in the tank at time t,
- V_0 be the volume of fluid in the tank at time $t = 0$, i.e., $V_0 = V(0)$

A Classic Mixing Problem Illustrated

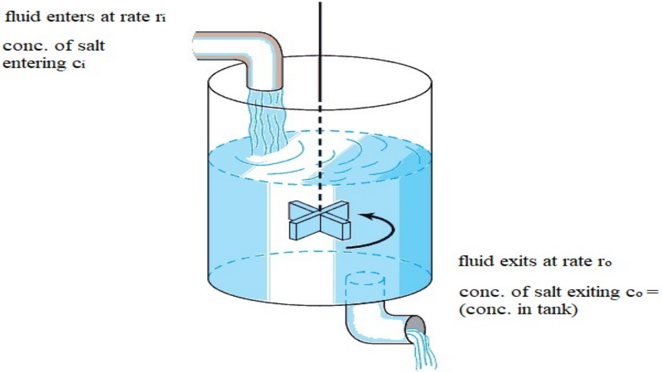


Figure: Values for c_i, r_i, and r_o are given in the problem statement. The well mixed assumption means that c_o will match the concentration in the tank.

This means that c_o is **NOT constant**! It depends on time through both A and V.

Building an Equation

What is the rate of change of the mass of the salt?

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array} \right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array} \right)$$

where

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i \cdot c_i.$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o \cdot c_o.$$

The parameters r_i, c_i, and r_o are part of the problem statement. We must determine c_o.

Building an Equation

By the well mixed solution assumption, the concentration of salt in the out-flowing fluid matches the concentration in the tank. That is,

$$c_o = \frac{\text{total salt}}{\text{total volume}} \quad = \quad \frac{A(t)}{V(t)} \quad = \quad \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

Note that the volume

$$V(t) = \textit{initial volume} + \textit{rate in} \times \textit{time} - \textit{rate out} \times \textit{time}.$$

If $r_i = r_o$, then $V(t) = V(0)$ a constant.

Pulling this together, the amount A satisfies the first order linear ODE

$$\boxed{\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.}$$

Solve the Mixing Problem

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$V(0) = 500$ gal

$r_i = 5 \frac{\text{gal}}{\text{min}}$

$c_i = 2 \frac{\text{lb}}{\text{gal}}$

$r_o = 5 \frac{\text{gal}}{\text{min}}$

$$c_o = \frac{A}{V} = \frac{A(t)}{V(0) + (r_i - r_o)\,t} \frac{\text{lb}}{\text{gal}}$$

$$= \frac{A}{500} \frac{\text{lb}}{\text{gal}}$$

$$\frac{dA}{dt} = 5 \frac{\text{gal}}{\text{min}} \cdot 2 \frac{\text{lb}}{\text{gal}} - 5 \frac{\text{gal}}{\text{min}} \cdot \frac{A}{500} \frac{\text{lb}}{\text{gal}}$$

$$\frac{dA}{dt} = 10 - \frac{1}{100}A \quad , \quad A(0) = 0$$ pure water

To separate variables, write as

$$\frac{dA}{dt} = \frac{1}{100}(1000 - A)$$

Using an integrating factor

$$\frac{dA}{dt} + \frac{1}{100}A = 10$$

$$P(t) = \frac{1}{100} \quad , \quad \mu = e^{\int \frac{1}{100}dt} = e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t}A\right) = 10\,e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) dt = \int 10\, e^{\frac{1}{100}t}\, dt$$

$$e^{\frac{1}{100}t} A = 10(100)\, e^{\frac{1}{100}t} + C$$

$$A = 1000 + C e^{\frac{-1}{100}t}$$

Apply $A(0) = 0$, $A(0) = 1000 + Ce^{0} = 0$

$$C = -1000$$

The amount of salt in pounds at time t in minutes is

$$A(t) = 1000 - 1000e^{\frac{-1}{100}t}$$

The concentration @ time t

is $\dfrac{A(t)}{V(t)}$

After 5 minutes it is

$$\frac{A(5)}{V(5)} = \frac{1000 - 1000e^{\frac{-1}{100}(5)}}{500} \frac{lb}{gal}$$

$$\approx 0.0975 \ lb/gal$$

A Nonlinear Modeling Problem

The last model we will consider is a nonlinear population model. It can account for reproduction and environmental limitations. Let's consider it through an example.

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

rate of change of P is $\frac{dP}{dt}$

population. $P(t)$

Difference between M and population

$M - P$.

$$\frac{dP}{dt} = kP(M-P).$$

k is some constant.

[1]The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

Logistic Growth Model

The equation $\frac{dP}{dt} = kP(M - P)$, where $k, M > 0$ is called a **logistic growth equation**.

Suppose the intial population $P(0) = P_0$. Solve the resulting initial value problem. Show that if $P_0 > 0$, the population tends to the carrying capacity M.

The ODE is separable and Bernoulli.

We obtained a 1-parameter family of solutions last time $P(t) = \frac{M}{1 + Ae^{-kMt}}$.

We need to apply the initial condition.

Logistic Growth: $P'(t) = kP(M - P)$ $P(0) = P_0$

Long Time Solution of Logistic Equation

$$\frac{dP}{dt} = kP(M - P) = -kP^2 + kMP.$$

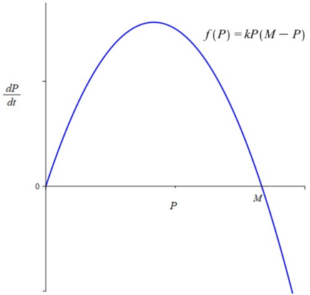


Figure: Plot of P versus $\frac{dP}{dt}$. Note that $\frac{dP}{dt} > 0$ if $0 < P < M$ and $\frac{dP}{dt} < 0$ if $P > M$.

Expected Long Time Solutions

Suppose we modify the logistic equation based on the assumption that the fish will only breed successfully if the population is above some minimum threshhold N where $0 < N < M$. The new model is

$$\frac{dP}{dt} = kP(M - P)(P - N).$$

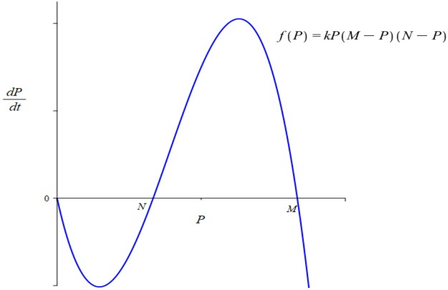


Figure: Plot of P versus $\frac{dP}{dt}$ for the modified model. There are two long-time scenarios, extinction and achieving carrying capacity.

Expected Long Time Solutions

Use the given plot of $F(P) = kP(M - P)(P - N)$ to determine the long time solution of $\frac{dP}{dt} = F(P)$ if (a) $0 < P(0) < N$, (b) $N < P(0) < M$ or (c) $P(0) > M$.

Qualitative Analysis

Autonomous Equation

The differential equation $\frac{dy}{dt} = f(t, y(t))$ is called **autonomous** if the right hand side does not depend explicitly on t—i.e., an autonomous equation has the form

$$\frac{dy}{dt} = F(y).$$

Equilibrium Solutions

Equilibrium Solutions

If y_0 is a value such that $F(y_0) = 0$, then the constant function $y(t) = y_0$ is called an **equilibrium** solution (or equilibrium point) of the autonomous differential equation $\frac{dy}{dt} = F(y)$.

Note: If $y(0) = y_0$ and y_0 is an equilibrium solution, then $y(t) = y_0$ is a constant solution.

Question: What if $y(0)$ is not an equilibrium value, but is close to an equilibrium value? What can we expect from the solution?

Stability of Equilibrium Solutions

In general, we may classify an equilibrium solution of a given autononous ODE as being

- **unstable**: solutions close, but not exact, will tend away from the equilibrium value,
- **stable**: solutions close, but not exact, will tend towards the equilibrium value[2], or
- **semi-stable**: solutions close, but not exact, may tend towards or away from the equilibrium value depending on whether the solution is greater than or less than the equilibrium value.

Note: There are more detailed notions of stability, so there's more to the story. But we'll consider the above definitions here.

[2] This is more accurately referred to as asymptotically stable

Determining Stability of an Equilibrium Solution

To determine the nature of an equilibrium solution y_0 for an ODE $y' = F(y)$, we can analyze the sign of F in the neighborhood of y_0. Suppose F is continuous on an open interval about y_0.

- If F changes sign from positive (+) to negative (-) as y passes through y_0 (from left to right), then y_0 is a stable equilibrium.
- If F changes signs from negative (-) to positive (+) as y passes through y_0, then then y_0 is an unstable equilibrium.
- If F doesn't change signs, then y_0 is semi-stable.

If this reminds you of a *derivative test*, there's a good reason for that. Fortunately, it's easy to visualize the cases if you can obtain even a crude drawing of the graph of F.

Example

Consider the IVP

$$y' = 2(y+1)(2-y)^2(y-3), \quad y(0) = k.$$

Determine the long time behavior, $\lim_{t\to\infty} y(t)$, if

(a) $k = -2$,	(b) $k = 0$,	(c) $k = 1$,
(d) $k = 2$,	(e) $k = 2.5$,	(f) $k = 4$.

Models Derived in this Section

We have several models involving first order ODEs.

Exponential Growth/Decay $$\frac{dP}{dt} = kP$$

RC-Series Circuit $$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

LR-Series Circuit $$L\frac{di}{dt} + Ri = E(t)$$

Classical Mixing $$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A(t)}{V(0) + (r_i - r_o)t}$$

Logistic Growth $$\frac{dP}{dt} = kP(M - P)$$