September 9 Math 2306 sec. 53 Fall 2024

Section 5: First Order Equations: Models and Applications

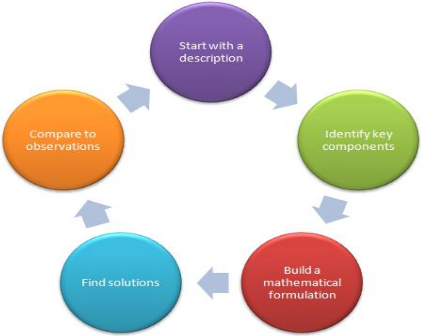

Figure: Mathematical Models give Rise to Differential Equations

In this section, we will consider select models involving first order ODEs. Let's see the process in action.

RC and LR Series Circuits

We want to track the charge on a capacitor (RC circuit) or the current in a circuit (LR circuit). Recall that current i is rate of change of charge q. These are functions of time.

The voltage across each type of element is shown below:

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Table: The potential drop across various elements.

Kirchhoff's Law

Kirchhoff's Law

Kirchhoff's Law states that:
The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force. We can use this to arrive at a differential equation for the charge $q(t)$ in an RC circuit or the current $i(t)$ in and LR circuit.

Both of these result in a first order linear differential equation.

RC Series Circuit

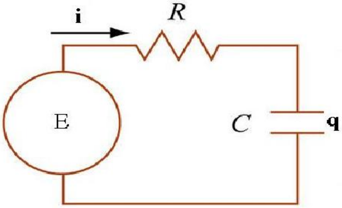

Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

drop across resistor	$+$	drop across capacitor	$=$	applied force
$R\frac{dq}{dt}$	$+$	$\frac{1}{C}q$	$=$	$E(t)$

$$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t)}$$

If $q(0) = q_0$, the IVP can be solved to find $q(t)$ for all $t > 0$.

LR Series Circuit

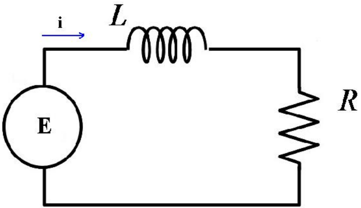

Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

$$\begin{array}{ccccc} \text{drop across inductor} & + & \text{drop across resistor} & = & \text{applied force} \\ L\frac{di}{dt} & + & Ri & = & E(t) \end{array}$$

$$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

If $i(0) = i_0$, the IVP can be solved to find $i(t)$ for all $t > 0$.

Summary of First Order Circuit Models

Before considering an example, let's summarize our two circuit models.

The charge $q(t)$ at time t on the capacitor in an RC-series circuit with resistance R ohm, capacitance C farads, and applied voltage $E(t)$ volts satisfies

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t), \quad q(0) = q_0$$

where q_0 is the initial charge on the capacitor.

The current $i(t)$ at time t in an LR-series circuit with resistance R ohm, inductance L henries, and applied voltage $E(t)$ volts satisfies

$$L\frac{di}{dt} + Ri = E(t), \quad i(0) = i_0$$

where i_0 is the initial current in the circuit.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$R = 1000\ \Omega$

$C = 5 \cdot 10^{-6}\ f$

$E(t) = 200\ V$

$$1000\frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}} q = 200 \qquad i(0) = q'(0) = 0.4\ A$$

$$\frac{1}{5 \cdot 10^{-6}} = \frac{10^6}{5} = 2 \cdot 10^5$$

Put in standard form

$$\frac{dq}{dt} + \frac{2 \cdot 10^5}{1000} q = \frac{200}{1000}$$

$$\frac{dq}{dt} + 200q = 0.2$$

Linear w/ $p(t) = 200$, $\mu = e^{\int 200 dt} = e^{200t}$

$$\frac{d}{dt}\left(e^{200t} q\right) = 0.2 e^{200t}$$

$$\int \frac{d}{dt}\left(e^{200t} q\right) dt = \int 0.2 e^{200t} dt$$

$$e^{200t} \cdot q = \frac{0.2}{200} e^{200t} + k$$

$$q = \frac{0.001 e^{200t} + k}{e^{200t}} = 0.001 + k e^{-200t}$$

Apply $q'(0) = 0.4$, $q'(t) = 0 - 200 k e^{-200t}$

$$q'(0) = -200 k e^{0} = 0.4$$

$$k = \frac{0.4}{-200} = -0.002$$

The charge on the capacitor is

$$q(t) = 0.001 - 0.002 e^{-200t}$$

The long term charge

$$\lim_{t \to \infty} q(t) = \lim_{t \to \infty} \left(0.001 - 0.002\, e^{-200t}\right)$$

$$= 0.001 \quad C$$

A Classic Mixing Problem

Classical mixing involves tracking the mass of some substance in a composite mixture. Examples include

- salt in a salt-water mixture,
- ethanol in an ethanol-gasoline mixture,
- polutant in a volume of water.

Let's look at a specific problem and build a model that can be used in general. First, a visual.

A Classic Mixing Problem

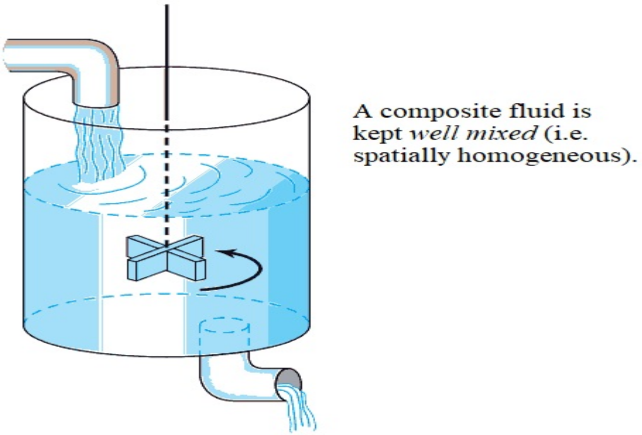

Figure: We wish to track the amount of some substance in a composite mixture such as salt and water, gas and ethanol, polutant and water, etc. Fluid may flow in and out of the composition, and we assume instant mixing so that the mass of some substance is dependent on time, but not on space.

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5 gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

In order to answer such a question, we need to convert the problem statement into a mathematical one.

Some Notation

In addition to the amount of salt, $A(t)$, at time t we have several variables or parameters. Let

- r_i be the rate at which fluid enters the tank (rate in),
- r_o be the rate at which fluid leaves the tank (rate out),
- c_i be the concentration of substance (salt) in the in-flowing fluid (concentration in),
- c_o be the concentration of substance (salt) in the out-flowing fluid (concentration out),
- $V(t)$ be the total volume of fluid in the tank at time t,
- V_0 be the volume of fluid in the tank at time $t = 0$, i.e., $V_0 = V(0)$

A Classic Mixing Problem Illustrated

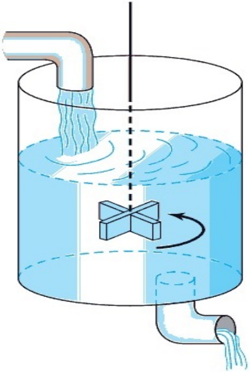

Figure: Values for c_i, r_i, and r_o are given in the problem statement. The well mixed assumption means that c_o will match the concentration in the tank.

This means that c_o is **NOT constant**! It depends on time through both A and V.

Building an Equation

What is the rate of change of the mass of the salt?

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array} \right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array} \right)$$

where

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i \cdot c_i.$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o \cdot c_o.$$

The parameters r_i, c_i, and r_o are part of the problem statement. We must determine c_o.

Building an Equation

By the well mixed solution assumption, the concentration of salt in the out-flowing fluid matches the concentration in the tank. That is,

$$c_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

Note that the volume

$$V(t) = \textit{initial volume} + \textit{rate in} \times \textit{time} - \textit{rate out} \times \textit{time}.$$

If $r_i = r_o$, then $V(t) = V(0)$ a constant.

Pulling this together, the amount A satisfies the first order linear ODE

$$\boxed{\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.}$$

Solve the Mixing Problem

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$V(0) = 500$ gal

$c_i = 2 \frac{\text{lb}}{\text{gal}}$

$r_i = 5 \frac{\text{gal}}{\text{min}}$

$r_o = 5 \frac{\text{gal}}{\text{min}}$

$V(t) = V(0) + (r_i - r_o)t$

$= 500 + (5-5)t$ gal

$= 500$ gal

$c_o = \frac{A(t)}{500}$ lb/gal

$$\frac{dA}{dt} = r_i c_i - r_o \frac{A}{V} = 2 \frac{\text{lb}}{\text{gal}} \left(5 \frac{\text{gal}}{\text{min}}\right) - 5 \frac{\text{gal}}{\text{min}} \left(\frac{A \text{ lb}}{500 \text{ gal}}\right)$$

$$\frac{dA}{dt} = 10 - \frac{1}{100} A \quad , \quad A(0) = 0$$

pure water

To separate variables write

$$\frac{dA}{dt} = \frac{1}{100} (1000 - A)$$

Using an integrating factor

$$\frac{dA}{dt} + \frac{1}{100} A = 10$$

$$P(t) = \frac{1}{100} \quad , \quad \mu = e^{\int \frac{1}{100} dt} = e^{\frac{1}{100} t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) = 10\, e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) dt = \int 10\, e^{\frac{1}{100}t}\, dt$$

$$e^{\frac{1}{100}t} A = 10(100)\, e^{\frac{1}{100}t} + C$$

$$A = 1000 + C\, e^{-\frac{1}{100}t}$$

Apply $A(0) = 0$,

$$A(0) = 1000 + C\, e^{0} = 0$$

$\Rightarrow \quad C = -1000$

The amount of salt is

$$A(t) = 1000 - 1000\, e^{-\frac{1}{100}t} \text{ lbs}$$

The concentration when $t = 5$ is

$$\frac{A(5)}{V(5)} = \frac{1000 - 1000\, e^{-\frac{1}{100}(5)}}{500} \approx 0.0975 \ \frac{\text{lb}}{\text{gal}}$$

A Nonlinear Modeling Problem

The last model we will consider is a nonlinear population model. It can account for reproduction and environmental limitations. Let's consider it through an example.

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

P current population,

M - P, difference between carrying capacity and P.

$\frac{dP}{dt}$ - rate of population change

$$\frac{dP}{dt} = kP(M-P),$$

k is some constant.

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

Logistic Growth Model

The equation $\frac{dP}{dt} = kP(M - P)$, where $k, M > 0$ is called a **logistic growth equation**.

Suppose the intial population $P(0) = P_0$. Solve the resulting initial value problem. Show that if $P_0 > 0$, the population tends to the carrying capacity M.

We solved this separable / Bernoulli ODE and found that

$$P(t) = \frac{M}{1 + Ae^{-kMt}}$$