Some Calculus Related Properties of Exponentials

Function Basics: Some basic properties of the functions e^x and $\ln(x)$.

Function	Domain	Range	Limits	Key points
e^x	$(-\infty, \infty)$	$(0, \infty)$	$\lim_{x\to-\infty} e^x = 0$ $\lim_{x\to\infty} e^x = \infty$	$e^0 = 1$
$\ln(x)$	$(0, \infty)$	$(-\infty, \infty)$	$\lim_{x\to 0^+} \ln(x) = -\infty$ $\lim_{x\to\infty} \ln(x) = \infty$	$\ln(1) = 0$

Basic Properties: For each expression below, we assume that the values are well defined. For example, if we write $\ln(a)$ it is assumed that $a > 0$. The basic properties of exponentials and logarithms include:

(a) $e^{x+y} = e^x e^y$ (b) $e^{x-y} = \dfrac{e^x}{e^y}$ (c) $(e^x)^y = e^{xy}$

(d) $\ln(ab) = \ln(a) + \ln(b)$ (e) $\ln\left(\dfrac{a}{b}\right) = \ln(a) - \ln(b)$ (f) $\ln(a^r) = r\ln(a)$

(g) $e^{\ln x} = x$ (h) $\ln(e^x) = x$

Calculus: The following derivative and antiderivative properties hold. It is assumed that $f(x)$ is some differentiable function, and a is a nonzero constant.

(a) $\dfrac{d}{dx}e^x = e^x$ (b) $\dfrac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$

(c) $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$ (d) $\dfrac{d}{dx}\ln(f(x)) = \dfrac{f'(x)}{f(x)}$

(e) $\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$ (f) $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

(g) $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$ (g) $\displaystyle\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$

Caution: Don't confuse the power rule with the antiderivative of $\frac{1}{x}$. That is,

$$\int \frac{1}{x}\,dx = \ln|x| + C, \quad \text{but} \quad \int \frac{1}{x^2}\,dx = -\frac{1}{x} + C \quad \text{and} \quad \int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C.$$

The last two are power rules. Just because there is a ratio does not mean that the integral is a logarithm. The power rule is

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for all } n \textbf{ except } -1; \quad \int x^{-1}\,dx = \ln|x| + C.$$

Combining Properties: We can combine rules of exponents and logarithms using that these are inverse operations. Some examples include

(a) $(e^x)^2 = e^{2x}$

(b) $(e^x)^{1/2} = e^{x/2}$

(c) $(e^x)^k = e^{kx}$ for any number k

(d) $e^{-\ln(x)} = e^{\ln(x^{-1})} = x^{-1}$

(e) $e^{2\ln(x)} = e^{\ln(x^2)} = x^2$

(f) $e^{k\ln(x)} = e^{\ln(x^k)} = x^k$ for any number k

(g) $e^{x+\ln(x)} = e^x\left(e^{\ln(x)}\right) = e^x(x) = xe^x$

(h) $e^{x+\ln(x+1)} = e^x\left(e^{\ln(x+1)}\right) = e^x(x+1) = (x+1)e^x$

(i) Here's a complicated example:

$$\frac{e^{x+\ln(x-1)}}{(e^x)^2} = \frac{e^x\left(e^{\ln(x-1)}\right)}{e^{2x}} = \frac{(x-1)e^x}{e^{2x}} = (x-1)e^{x-2x} = (x-1)e^{-x}$$