A Workbook to Accompany MATH 2306

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Introduction: This book contains practice problems and exercises to accompany the introductory course in Differential Equation, KSU's MATH 2306. Many of the exercises here are of my own creation—most are the standard fair one would find in any commercial or open access introductory textbook. A few of the exercises are reproduced from open educational resources or are inspired by a commercial text. Where appropriate, I have included citations so that the interested reader can consult the appropriate source. Occassionally, asides are shown in blue to alert the reader to a significant fact or connection.

Errata: While I hope this material is free of errors, there are likely typos or mathematical mistakes. Please report errors to me at lritter@kennesaw.edu.

Contents

Chapter 1

Exercises

The activities in this text include

- Building a glossary of significant terms and phrases,
- An introductory *get acquainted problem* for each section, and
- Regular exercises and application problems.

Occasionally, a particularly challenging problem is included to allow you to explore a concept in greater depth. These are identified by the marker **(Challenge Problem)** .

Most sections start with a list of **Glossary Items**. These are terms and phrases that are significant to the study of differential equations. I am recommending that you start to compile your own glossary and add to it as the course progresses. Appropriate definitions can be found in the lecture notes or by conducting a simple search through the course text (slides) or of the web. Writing down these terms (phrases) along with their meanings will help you to gain familiarity with the language commonly used in the study of differential equations so that you can both read and write confidently.

The **Get Acquainted Problems** should be completed on the day in which the section is introduced in class. Even if you do not have time to complete much of the assigned problems in a given section, this problem should be worked in full on that first day. In general, these problems are exemplars of the concepts for the corresponding section. Working this problem, prior to going to sleep for the first time, may help to move the terms and concepts from working memory (short term) into long term memory.

Solutions to all problems are provided in the *back* of this book. At the beginning of the problem set for each section, there is a hyperlink to the part of the book containing that section's solutions. The section with the corresponding solutions begins with a link to return to the problem set. Whether you can use these links may depend on the pdf viewer that you use.

Introduction and 1^{st} Order Equations

1.1 Section 1: Concepts and Terminology

Write the definition of each term (phrase).

Glossary Item 1. Differential Equation

Glossary Item 2. Order of a differential equation

Glossary Item 3. Linear ordinary differential equation

Glossary Item 4. Explicit solution

Glossary Item 5. Implicit solution

Glossary Item 6. Domain of definition

Glossary Item 7. Parameter

Glossary Item 8. n-Parameter family of solutions

Glossary Item 9. Particular solution

Glossary Item 10. Trivial solution

Go to Solutions 2.1. ⟵ That number, 2.1, is a hyperlink to the solutions.

Get Acquainted Problem 1. Consider the differential equation

$$x^2y'' + xy' - y = \ln(x), \quad x > 0. \tag{1.1}$$

Let $\phi(x) = x + \frac{1}{x} - \ln(x)$ for $x > 0$. Take the following steps to show that ϕ is a solution to the equation on the interval $(0, \infty)$.

(a) Set $y = \phi = x + \frac{1}{x} - \ln(x)$. Using the derivative rules from calculus, find y' and y''.

(b) Using y and the derivatives you just computed, simplify the expression

$$x^2y'' + xy' - y$$

as much as possible. (That's the left hand side of equation (1.1).)

(c) Now compare your result to the right hand side of equation (1.1), namely $\ln x$. Are they the same for all $x > 0$?

(d) How does this demonstrate that ϕ is a solution of equation (1.1) on the indicated interval?

Problem 1. For each differential equation, identify (i) the independent variable(s), (ii) the dependent variable(s), (iii) the order, (iv) and whether it is an ordinary or a partial differential equation.

(a) $\frac{d^3y}{dx^3} + xy^4 = \cos(2x)$

(b) $\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$

(c) $t^2 \frac{d^2x}{dt^2} + 3t\frac{dx}{dt} - 15x = 0$

(d) $\frac{dy}{dt} - 4\frac{d^2x}{dt^2} = 0$

Problem 2. For each of the differential equations shown, determine if the equation is Linear or Nonlinear.

(a) $t^2 \frac{d^2x}{dt^2} + 3t\frac{dx}{dt} - 15x = 0$

(b) $t^2 \frac{d^2x}{dt^2} + 3x\frac{dx}{dt} - 15x = 0$

(c) $\frac{d^2y}{dx^2} + xy^4 = \cos(2x)$

(d) $\frac{d^2y}{dx^2} + x\frac{d^4y}{dx^4} = \cos(2x)$

(e) $\dfrac{dr}{dt} = \dfrac{\beta}{3\rho}\left(\dfrac{3\alpha}{\beta} - r\right)$ (assume α, β and ρ are constant)

Problem 3. Use your knowledge of basic calculus to find a solution to each differential equation. Give a possible domain of definition for your solution.

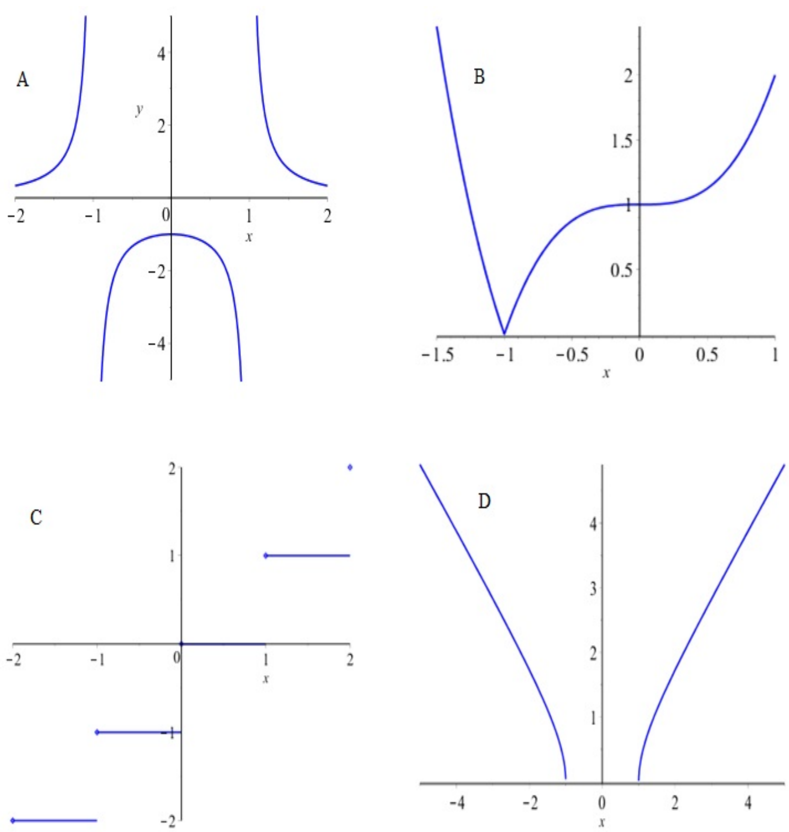
(a) $x\dfrac{dy}{dx} = 1$

(b) $y'' = 0$

(c) $\dfrac{dx}{dt} = \dfrac{1}{\sqrt{1-t^2}}$

(d) $\dfrac{d^3y}{dx^3} = x + e^{2x}$

Problem 4. For each plot, determine if it could be the graph of a solution to a differential equation. Provide some justification for your conclusion.



Problem 5. Verify that each function ϕ or relation $G(x, y) = C$ defines a solution to the indicated differential equation.

(a) $\phi(t) = \cot(2t), \quad y^2 + 1 = -\frac{1}{2}y'$

(b) $\ln(xy) - x = 0, \quad y + xy' = xy$

(c) $\phi(x) = c_1 e^x + c_2 e^{-3x}, \quad y'' + 2y' - 3y = 0$

(d) $x^2 y - y^2 x + x + 4y = C, \quad (2xy - y^2 + 1)\,dx + (x^2 - 2xy + 4)\,dy = 0$

Problem 6. Find all possible values of the constant m such that $y = x^m$ solves the differential equation

(a) $x^2 \dfrac{d^2y}{dx^2} - 6x\dfrac{dy}{dx} + 12y = 0$

(b) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} - y = 0$

Problem 7. Find all possible values of the constant m such that $y = e^{mx}$ solves the differential equation

(a) $y'' - 2y' - 8y = 0$

(b) $y'' - 6y' + 7y = 0$

Problem 8. Consider the functions $a_0(x) = -1$, $a_1(x) = x$ and $a_2(x) = x^2$ for $x > 0$. Suppose y is any function that is twice continuously differentiable on $(0, \infty)$. Define an operator L by its *action* on y

$$Ly = a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y.$$

(a) Evaluate Ly if $y = x$

(b) Evaluate Ly if $y = x^2$

(c) Evaluate Ly if $y = x^{-1}$

Note that an equation of the form $Ly = f(x)$ is a linear ODE.

Problem 9. **(Challenge Problem)** Suppose a_0, a_1, and a_2 are known functions defined on some interval I, and let the operator L be defined by its *action* on a twice continuously differentiable function y

$$Ly = a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y.$$

Show that

(a) If y and z are twice differentiable on I, then $L(y+z) = Ly + Lz$.

(b) For any constant c, $L(cy) = cLy$

Now consider a new operator N defined by

$$Ny = y\frac{dy}{dx}$$

Show that

(a) In general (i.e. we can find functions y and z such that), $N(y+z) \neq Ny + Nz$.

(b) If c is a constant different from 0 or 1, and y is not a constant function, then $N(cy) \neq cNy$

Note that a differential equation $Ly = F(x)$ would be a *linear* equation whereas the equation $Ny = F(x)$ would be nonlinear.

1.2 Section 2: Initial Value Problems

Write the definition of each term (phrase).

Glossary Item 11. Initial Condition

Glossary Item 12. Initial Value Problem

Glossary Item 13. Existence and Uniqueness (of a solution)

Go to Solutions 2.2.

Get Acquainted Problem 2. Imagine a real number line marked out in inches, and let $s(t)$ be the position of an ant at time t in seconds that is walking on this line. Suppose the ant's velocity is known to be

$$v = \frac{ds}{dt} = \frac{1}{\pi^2}t - \sin(t).$$

(a) Find all such functions $s(t)$ using a regular indefinite integral.

(b) Suppose that at time $t = 0$, the ant's position is at zero on the number line. Determine the ant's position for all $t > 0$.

(c) Where is the ant one second later when $t = 1$?

Problem 1. Solve each initial value problem using your knowledge from Calculus.

(a) $\dfrac{dy}{dx} = xe^x, \quad y(0) = 2$

(b) $\dfrac{d^2y}{dx^2} = \cos x \quad y(0) = 1, \quad y'(0) = -1$

(c) $\dfrac{dx}{dt} = \dfrac{1}{\sqrt{1-t^2}}, \quad x\left(\dfrac{1}{2}\right) = 0$

Problem 2. It can be shown that all solutions of the differential equation $x^2y'' + 2xy' - 6y = 0$ on the interval $(0, \infty)$ are in the family

$$y = c_1x^2 + c_2x^{-3}.$$

Solve each IVP (initial value problem).

(a) $x^2y'' + 2xy' - 6y = 0, \quad y(1) = 4, \quad y'(1) = 2$

(b) $x^2y'' + 2xy' - 6y = 0, \quad y(2) = 1, \quad y'(2) = 0$

Problem 3. Consider the first order initial value problem $\dfrac{dy}{dt} = 1 - y$ subject to $y(0) = y_0$.

(a) Find the constant solution to the initial value problem $\dfrac{dy}{dt} = 1 - y$ subject to $y(0) = 1$. (Hint: start with the initial condition.)

(b) Can you come up with an argument for why $\lim_{t\to\infty} y(t) = 1$ is expected no matter what the value of y_0 is? (Hint: think about what the sign of a derivative tells you about a function.)

Problem 4. Consider the third order equation

$$y''' + 2y'' + y' = 0 \quad \text{for } -\infty < x < \infty.$$

(a) Verify that each of the functions $y_1 = e^{-x}$, $y_2 = xe^{-x}$ and $y_3 = 1$ solve this ODE.

(b) Show that the function $y = c_1y_1 + c_2y_2 + c_3y_3$ is a solution of the ODE for any constants c_1, c_2, c_3. (This is called the *general solution*.)

(c) Solve the IVP $\quad y''' + 2y'' + y' = 0, \quad y(0) = 6, \quad y'(0) = -3, \quad y''(0) = 4.$

(d) It is obvious that $Y = c_1e^{-x} + c_2xe^{-x}$ is a solution to the ODE $Y''' + 2Y'' + Y' = 0$ for any choice of constants c_1 and c_2. (Why is this now obvious?) Show that it is not possible to find constants c_1 and c_2 such that $Y = c_1e^{-x} + c_2xe^{-x}$ solves the IVP

$$Y''' + 2Y'' + Y' = 0 \qquad Y(0) = 1, \quad Y'(0) = 2, \quad Y''(0) = 3$$

This suggests that we have to be concerned with finding *all possible solutions* as opposed to just finding a solution when dealing with ODEs.

Problem 5. Consider the problem given by the equation and conditions

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \quad \text{for } a < x < b$$

$$\text{subject to} \quad y(a) = y_0, \quad \text{and} \quad y(b) = y_1.$$

This problem is an example of a second order, linear **boundary value problem** (BVP). Despite looking similar to an IVP, our existence and uniqueness results do not generally hold for this type of problem. To see this, consider the BVP

$$\frac{d^2y}{dx^2} + 4y = 0 \quad \text{for} \quad 0 < x < b$$

$$\text{subject to} \quad y(0) = y_0, \quad \text{and} \quad y(b) = y_1.$$

It can be shown that all solutions to this ODE are members of the family[1] of functions $y = c_1\cos(2x) + c_2\sin(2x)$.

[1] We'll see how to obtain this family when we get to section 8.

(a) Show that if $b = \pi$, $y_0 = 0$ and $y_1 = 0$, the BVP has infinitely many solutions.

(b) Show that if $b = \pi$, $y_0 = 0$ and $y_1 = 1$, the BVP has no solutions.

(c) Show that if $b = \frac{\pi}{4}$, $y_0 = 0$ and $y_1 = 0$, the BVP has exactly one solution.

Problem 6. Consider the initial value problem

$$\frac{dy}{dx} = x^2y^2, \quad y(0) = 1$$

(a) Use Euler's method with a step size $h = 0.1$ to approximate $y(0.2)$. Keep five decimal digits.

(b) Use Euler's method with a step size of $h = 0.05$ to approximate $y(0.2)$. Keep five decimal digits.

(The true value is $y(0.2) = \frac{375}{374} \approx 1.00267$.)

Problem 7. Consider the initial value problem

$$\frac{dy}{dx} + 4y - 2x = 0, \quad y(1) = 0$$

(a) Use Euler's method with a step size $h = 0.1$ to approximate $y(1.2)$. Keep five decimal digits.

(b) Use Euler's method with a step size of $h = 0.05$ to approximate $y(1.2)$. Keep five decimal digits.

(The exact value is $y(1.2) = 0.475 - 0.375e^{-0.8} \approx 0.306502$.)

Problem 8. Consider the initial value problem

$$\frac{dy}{dx} = x - y, \quad y(1) = k.$$

Suppose that Euler's method is used with a step size of $h = \frac{1}{2}$, and after two steps, the approximation $y(2) \approx y_2 = 3$ is obtained. Determine the value of k.

Problem 9. **(Challenge Problem)** Consider the IVP $\dfrac{dy}{dt} = 3y^{2/3}, \quad y(0) = 0.$

(a) Show that $y = t^3$ is a solution of this IVP.

(b) Find a constant solution $y(t) = K$ for $-\infty < t < \infty$ that solves this IVP.

(c) Show that for any constant $c \geq 0$, the functions $y_c(t) = \begin{cases} 0, & t < c \\ (t-c)^3, & t \geq c \end{cases}$ solve the IVP. (Take care when differentiating y_c at c.)

(d) Is the constant function you found in (b) a member of the family in (c)?

1.3 Section 3: First Order Equations: Separation of Variables

Write the definition of each term (phrase).

Glossary Item 14. First Order Separable Differential Equation

Go to solutions 2.3.

Get Acquainted Problem 3. Consider the initial value problem

$$\frac{dy}{dx} = x^2\sqrt{y}, \quad y(0) = 1.$$

(a) Rewrite the differential equation by dividing both sides by $\sqrt{y}$ and multiplying both sides by the differential dx.

(b) Recall that the product $\frac{dy}{dx}\,dx$ is the differential dy. Integrate both sides of the equation you obtained in part (a); each side should be integrated with respect to only one variable.

(c) Apply the condition $y(0) = 1$ to find the particular solution to the IVP. Present the solution in two ways, implicitly and explicitly.

(d) What did we assume about the value of y in the first step when dividing by $\sqrt{y}$? Given the whole IVP, why did this seem like a reasonable assumption?

Problem 1. For each first order ODE, determine if the equation is separable. If separable, find all solutions.

(a) $\dfrac{dy}{dx} = xy$

(b) $y' + y^2 = 0$

(c) $\dfrac{dy}{dx} = x^2 + y^2$

(d) $\dfrac{dx}{dt} = xt + 2x + t + 2$

(e) $\dfrac{du}{dt} = (u^2 - 1)t$ (Adapted from [1])

(f) $\dfrac{dy}{dx} + xy = x$

(g) $\dfrac{dy}{dx} + xy = x^2$

Problem 2. Solve each initial value problem. Give an explicit solution where possible.

(a) $\dfrac{dy}{dx} = xy, \quad y(1) = 1$

(b) $\dfrac{dy}{dt} = \dfrac{y^2+1}{t^2+1}, \quad y(0) = 1$ (Adapted from [1])

(c) $\dfrac{dx}{dt} = \dfrac{3t^2}{x \ln x} \quad x(0) = e$

(d) $\dfrac{du}{dt} = (u^2 - 1)t, \quad u(0) = 0$ (Adapted from [1])

(e) $\dfrac{dy}{dt} = (\cos^2 y)(\cos^2 t), \quad y(0) = \dfrac{\pi}{4}$

Problem 3. Suppose M is a positive constant. Find an explicit solution to the IVP

$$\frac{dP}{dt} = P(M-P), \quad P(0) = P_0$$

where the initial value $P_0 > 0$. Determine the long time value of the solution, i.e. $\lim_{t\to\infty} P(t)$.
This IVP is an example of a *logistic* equation commonly used in population modeling.

Problem 4. Suppose α, β, ρ, and r_0 are positive constants. Solve the initial value problem.

$$\frac{dr}{dt} = \frac{\beta}{3\rho}\left(\frac{3\alpha}{\beta} - r\right), \quad r(0) = r_0.$$

Show that $r(t) \to \dfrac{3\alpha}{\beta}$ as $t \to \infty$.
You might recognize this as the spherical cell growth model from the introduction.

Problem 5. Verify that $y = y_0 + \displaystyle\int_0^x e^{-t^2}\,dt$ is a solution to the IVP

$$\frac{dy}{dx} = e^{-x^2}, \quad y(0) = y_0$$

The error function erf, which is related to the normal distribution, is defined for all real x by $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt$. The solution given in this exercise can be written as $y = y_0 + \dfrac{\sqrt{\pi}}{2}\operatorname{erf}(x)$.

Problem 6. **(Challenge Problem)** Verify that $y = y_0 \exp\left(\int_0^x e^{-t^2}\,dt\right)$ is a solution to the IVP

$$\frac{dy}{dx} = ye^{-x^2}, \quad y(0) = y_0$$

1.4 Section 4: First Order Equations: Linear & Special

Write the definition of each term (phrase).

Glossary Item 15. First order linear differential equation

Glossary Item 16. homogeneous First order linear differential equation

Glossary Item 17. Standard form (for first order linear ODE)

Glossary Item 18. Transient and steady states

Glossary Item 19. Integrating factor

Glossary Item 20. Bernoulli equation

Glossary Item 21. Differential Form

Glossary Item 22. Exact differential equation

Go to solutions 2.4

Get Acquainted Problem 4. Consider the first order linear equation

$$\frac{dy}{dx} + 2y = 18$$

(a) Multiply both sides of this equation by $\mu = e^{2x}$.

(b) Find the derivative $\frac{d}{dx}\left[e^{2x}y(x)\right]$ using the product and chain rules assuming y is some differentiable function of x

(c) Compare the derivative you found in (b) with the left side of the equation you wrote in (a). Use this to collapse your ODE into an equation that looks like

$$\frac{d}{dx}\left[\text{something}\right] = \text{something else}$$

(d) Integrate your equation, and isolate y.

(e) Substitute the solution y that you found back into the original differential equation to verify that it is indeed a solution (more accurately a *family of solutions*).

Problem 1. For each expression of the form $\frac{dy}{dx} + P(x)y$,

- find the integrating factor $\mu = e^{\int P(x)\,dx}$
- multiply the expression by μ

- find the derivative $\frac{d}{dx}(\mu(x)y(x))$, and compare this to $\mu\frac{dy}{dx} + \mu(x)P(x)y$.

(a) $\frac{dy}{dx} + (\tan x)y$

(b) $\frac{dy}{dx} - \frac{1}{x}y$

(c) $\frac{dy}{dx} - \frac{x}{x+1}y$

Problem 2. Find the general solution of each first order, linear ODE.

(a) $x\frac{dy}{dx} + y = 6x^2, \quad x > 0$

(b) $\frac{dy}{dx} + (\cot x)y = 1, \quad 0 < x < \pi$

(c) $\frac{dx}{dt} + x = e^{-t}\ln t, \quad t > 0$

(d) $\frac{du}{d\theta} + (\sec\theta)u = \cos\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(e) $t\frac{dy}{dt} - y = t^2\cos t, \quad t > 0$

Problem 3. Solve each initial value problem.

(a) $x\frac{dy}{dx} + y = 6x^2, \quad y(1) = 0$

(b) $\frac{dx}{dt} = -k(x - A), \quad x(0) = x_0$ assume k and A are positive constants

(c) $\frac{du}{d\theta} + (\sec\theta)u = \cos\theta, \quad u(0) = 0$

(d) $\rho\frac{dr}{dt} = \alpha - \frac{\beta}{3}r, \quad r(0) = r_0$ assume α, β, ρ and r_0 are positive constants

Problem 4. Assume that R_0, C_0, and E_0 are positive constants. Solve the initial value problem, and identify the transient and steady states.

$$R_0\frac{dq}{dt}+\frac{1}{C_0}q=E_0,\quad q(0)=0$$

Problem 5. Reverse engineer your own first order linear ODE. Pick a pair of functions μ and F (with μ reasonably easy to differentiate, and F reasonably easy to integrate), and start by setting it up as

$$\frac{d}{dx}(\mu(x)y)=F(x).$$

Then expand this to standard form. Provide your differential equation (in standard form), the general solution to your differential equation, and a valid domain of definition for your solution.

Problem 6. Express the solution of the IVP $\frac{dy}{dx}-2xy=\frac{2}{\sqrt{\pi}}$, $y(0)=0$ in terms of an integral.

Recalling the definition of the error function, erf, in section 1.3, you can express your solution in terms of the error function.

Problem 7. Consider the IVP with piece-wise defined right hand side

$$\frac{dy}{dt}+y=\begin{cases}1, & 0\le t<3\\ 0, & t\ge 3\end{cases}\qquad y(0)=0.$$

This problem can be solved in two stages with the solution constructed from these pieces.

(a) Solve the IVP $y_1'+y_1=1$ subject to $y_1(0)=0$. This will be the solution for $0\le t<3$.

(b) Now solve the new IVP $y_2'+y_2=0$ subject to $y_2(3)=y_1(3)$. This will be the solution for $t\ge 3$.

(c) Now express the solution to the original problem as the piecewise defined function

$$y(t)=\begin{cases}y_1(t), & 0\le t<3\\ y_2(t), & t\ge 3\end{cases}$$

and verify that this is continuous on $(0,\infty)$.

In section 16, we will have a method for dealing with such a problem that does not require us to consider solving for the pieces independently.

1.4.1 Some Special First Order Equations

Go to solutions 2.4.1

Problem 8. Show that each equation is exact and find the solutions.

(a) $(2xy^2 + 6)\,dx + (2x^2y - 7)\,dy = 0$

(b) $(e^x \cos y + e^y \cos x)\,dx + (e^y \sin x - e^x \sin y)\,dy = 0$

Problem 9. Solve each equation by using a special integrating factor that depends only on x or only on y.

(a) $(2y^3 + 3xy)\,dx + (3xy^2 + x^2)\,dy = 0$

(b) $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$

(c) $y^2\,dx + (e^x \sin y - 2y)\,dy = 0$

Problem 10. Consider the first order linear ODE $\dfrac{dy}{dx} + P(x)y = f(x)$.

(a) Multiply both sides by the differential dx, and using $dy = \frac{dy}{dx}\,dx$ rearrange the equation into the form $M(x,y)\,dx + N(x,y)\,dy = 0$.

(b) Show that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ (assuming P is not simply zero.)

(c) Show that $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ depends only on x and use this to obtain a special integrating factor $\mu(x)$.

(d) Compare this to the traditional integrating factor for the first order linear equation.

Exercises 11 and 12 were inspired by [3].

Problem 11. **(Challenge Problem)** Determine the functions $M(x,y)$ such that the given differential equation is exact.

$$M(x,y)\,dx + (xe^{xy} + 3x^2y^2 + 4)\,dy = 0$$

Problem 12. The given differential equation is nonlinear and nonseparable. Show that when written as a differential form, the resulting equation is exact. Use this to solve the initial value problem.

$$\frac{dy}{dx} = -\left(\frac{xy^2 + \cos x \sin x}{y(1+x^2)}\right), \quad y(0) = 3$$

Problem 13. Solve each Bernoulli equation.

(a) $xy' + y = x^2y^2$

(b) $x\frac{dy}{dx} + y = \frac{1}{y^2}$

(c) $\frac{dy}{dx} + y = y^{-1/2}$

Problem 14. Solve each initial value problem. [2]

(a) $xy' + y = x^4y^4, \quad y(1) = 1/2$

(b) $\frac{dy}{dx} - 2y = 2\sqrt{y}, \quad y(0) = 1$

1.5 Section 5: First Order Equations: Models and Applications

Write the definition of each term (phrase).

Glossary Item 23. Exponential growth or decay

Go to solutions 2.5

Get Acquainted Problem 5. (Adapted from [2]) A candy maker makes 300 pounds of candy per week. Let $Q(t)$ be the amount of candy in pounds at the time t in weeks. Suppose his large family eats candy at a rate of $\frac{Q(t)}{30}$ pounds per week.

(a) Write a differential equation for $Q(t)$ that accounts for a constant growth rate of 300 lb/wk and a variable decay rate of $Q/30$ lb/wk.

(b) Suppose the candy maker starts with 50 pounds of candy when $t = 0$. Solve the resulting initial value problem.

(c) What is the upper limit on the amount of candy the candy maker will ever have? (Look at $\lim_{t\to\infty} Q(t)$.)

Problem 1. An RC-series circuit has resistance R_0 and capacitance C_0. A constant electromotive force of E_0 volts is applied. If the initial charge on the capacitor $q(0) = 0$ coulombs, find the charge $q(t)$ on the capacitor at time t. Show that the long time charge on the capacitor is the product E_0C_0.

Problem 2. An LR-series circuit has inductance 0.5 h, resistance 10 ohms, and an applied force of $0.1e^{-t}$ volts. If the initial current $i(0) = 1$ amp, find the current $i(t)$.

Problem 3. A 400 gallon tank contains water into which 10 lbs of salt is dissolved. Salt water containing 3 lbs of salt per gallon is being pumped in at a rate of 4 gallons per minute, and the well mixed solution is being pumped out at the same rate. Let $A(t)$ be the number of lbs of salt in the tank at time t in minutes. Derive the initial value problem governing $A(t)$. Solve this IVP for A.

Problem 4. Suppose the solution in the last problem is being pumped out at the rate of 5 gallons per minute. Keeping everything else the same, derive the IVP governing A under this new condition. Solve this IVP for A. What is the largest time value for which your solution is physically feasible?

Problem 5. A 1000 gallon aquarium initially contains 500 gallons of pure water. A brine solution containing 0.5 lbs of salt per gallon is pumped in at a rate of 4 gallons per minute. The solution is kept well mixed, and the fluid is pumped out at the slower rate of 3 gallons per minute. Determine the concentration, C, of salt in the tank at the moment it starts to overflow.

Problem 6. A tank contains 100 liters of clean water. Water is flowing into the tank at a rate of 3 L/min, and this in-flowing water contains 10 gm/L of a toxic pollutant. The tank is kept well mixed, and the well mixed solution is pumped out at a rate of 4 L/min. Determine the mass (in grams) of pollutant in the tank at the moment that it is half empty.

Problem 7. A 500 gallon aquarium contains salt water with an initial concentration of 0.3 lbs per gallon. Fresh water is pumped in at the rate of 2 gallons per minute, and the well mixed solution is pumped out at the same rate. Identify the initial value problem modeling the amount of salt in pounds in the tank as a function of time in minutes. Determine the amount of salt, in pounds, for all $t > 0$. How many minutes does it take for the concentration in the tank to reach half it's initial level (i.e. 0.15 lbs/gal)?

Problem 8. An LR-series circuit has constant inductance and resistance L_0 and R_0, respectively. A constant electromotive force of E_0 is applied. Find the current $i(t)$ if $i(0) = i_0$ amp. Show that for any initial current, the long time current in the circuit is E_0/R_0.

Problem 9. (Adapted from [2]) A tank is empty at $t = 0$. Water is added to the tank at the rate of 10 gal/min, but at time $t > 0$ it leaks out at a rate (in gallons per minute) proportional to the number of gallons in the tank at that time t. Let $V(t)$ be the volume in gallons at time t in minutes.

(a) Let α be the proportionality constant described in the statement. What are the units of α?

(b) From the description, derive the differential equation and initial condition satisfied by V.

(c) Suppose the numerical value of α is $\frac{1}{2}$. Solve the IVP for V.

(d) Show that $V(t)$ is increasing for all $t > 0$. What is the minimum tank capacity necessary for this process to continue indefinitely?

Problem 10. (Adapted from [1]) Newton's law of cooling states that $\dfrac{dx}{dt} = -k(x - A)$ where x is the temperature of an object, t is time, A is the ambient temperature, and $k > 0$ is a constant. Usually, A is assumed to be constant, but suppose the ambient temperature is oscillating (for example between night and day temperatures). Let $A = A_0 + A_1 \cos(\omega t)$ for some constants A_0, A_1 and ω.

(a) Find the general solution of this linear equation.

(b) Consider the long time behavior of the solution. Do the initial conditions have much influence over this long time solution? (Explain your claim.)

Problem 11. [2] An epidemic spreads through a population at a rate proportional to the product of the number of people already infected and the number of people susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t, then

$$\frac{dI}{dt} = rI(S - I) \quad \text{where } r \text{ is a positive constant.}$$

Assuming that $I(0) = I_0$, find $I(t)$ for $t > 0$ and show that $I(t) \to S$ as $t \to \infty$.

Compare this to logistic equation problem from section 1.3. This model is for an epidemic, and it is expected that the entire susceptible population will eventually become infectected. A more interesting model might account for the possibility that infected individuals can recover returning them to the susceptible state. This leads to more interesting posibilities including complete eradication of disease. The following short video shows construction of such a differential equation along with some analysis of the possible outcomes. **Infection Video Link**

Problem 12. We know that a quantity P that satisfies $\frac{dP}{dt} = P$ experiences exponential growth. The equation $\frac{dP}{dt} = P^{1+\epsilon}$, for $\epsilon > 0$, is refered to as a *doomsday equation.* Solve the following IVP.

$$\frac{dP}{dt} = P^2, \quad P(0) = P_0 \quad \text{where, } P_0 > 0$$

By taking the limit $\lim_{t \to \frac{1}{P_0}^-} P(t)$, deduce the appropriateness of the *doomsday* label.

Problem 13. Consider the autonomous equation $\dfrac{dx}{dt} = 4 \sin\left(\dfrac{\pi x}{2}\right)$.

(a) Identify all equilibrium solutions of x such that $-5 < x < 5$.

(b) Classify each equilibrium solution found as being stable, unstable, or semi-stable.

(c) Determine the long time solution $\lim_{t \to \infty} x(t)$ if the equation is coupled with the initial condition

(i) $x(0) = -3.9$, (ii) $x(0) = -1$, (iii) $x(0) = 0$, (iv) $x(0) = 0.1$, (v) $x(0) = 3.9$.

Higher Order Linear Equations and Applications

1.6 Section 6: Linear Equations: Theory and Terminology

Write the definition of each term (phrase).

Glossary Item 24. HOMOGENEOUS AND NONHOMOGENEOUS LINEAR EQUATIONS

Glossary Item 25. PRINCIPLE OF SUPERPOSITION (for a homogeneous equation)

Glossary Item 26. PRINCIPLE OF SUPERPOSITION (for a particular solution)

Glossary Item 27. LINEAR DEPENDENCE/INDEPENDENCE

Glossary Item 28. FUNDAMENTAL SOLUTION SET

Glossary Item 29. GENERAL SOLUTION TO AN Nth ORDER LINEAR HOMOGENEOUS EQUATION

Glossary Item 30. GENERAL SOLUTION TO AN Nth ORDER LINEAR NONHOMOGENEOUS EQUATION

Go to solutions 2.6

Get Acquainted Problem 6. (Adapted from [2]) For the differential equation

$$y'' - 7y' + 10y = 0 \quad -\infty < x < \infty$$

(a) Verify that $y_1 = e^{2x}$ and $y_2 = e^{5x}$ form a fundamental solution set on $(-\infty, \infty)$. Remember to show that each function solves the ODE and that the functions are linearly independent.

(b) Write the general solution.

(c) Solve the IVP $y'' - 7y' + 10y = 0$ subject to $y(0) = -1$, $y'(0) = 1$.

(d) Solve the IVP $y'' - 7y' + 10y = 0$ subject to $y(0) = k_0$, $y'(0) = k_1$

Problem 1. Determine the Wronskian for each set of functions.

(a) $\{x^3, x^{-3}\}$

(b) $\{1, x, x^2\}$

(c) $\{e^{\alpha x}, e^{-\alpha x}\}$

(d) $\{e^{ax}\cos(bx), e^{ax}\sin(bx)\}$

(e) $\{x^m, x^n\}$

(f) $\{e^{ax}, xe^{ax}\}$

Problem 2. Determine whether the set of functions is linearly dependent or linearly independent on the given interval.

(a) $\{e^x, e^{x-3}\} \quad I = (-\infty, \infty)$

(b) $\{e^{\alpha x}, xe^{\alpha x}\} \quad I = (-\infty, \infty)$

(c) $\{\cos(2x), \sin^2(x), 1\} \quad I = (-\infty, \infty)$

(d) $\{e^{\alpha x}, e^{\beta x}\} \quad I = (-\infty, \infty)$

(e) $\{x^m, x^n\} \quad I = (0, \infty)$

Problem 3. Consider the differential equation $x^2y'' + xy' + y = 0$.

(a) Verify that $y_1 = \cos(\ln x)$ and $y_2 = \sin(\ln x)$ form a fundamental solution set on $(0, \infty)$

(b) Write the general solution.

(c) Solve the IVP $x^2y'' + xy' + y = 0$ subject to $y(1) = k_0$, $y'(1) = k_1$.

Problem 4. Suppose the pair of functions $\{y_1, y_2\}$ are a fundamental solution set to the second order, homogeneous differential equation

$$y'' + P(x)y' + Q(x)y = 0.$$

Let $z_1 = \frac{1}{2}(y_1 + y_2)$ and $z_2 = \frac{1}{2}(y_1 - y_2)$. Show that $\{z_1, z_2\}$ is also a fundamental solution set to this homogeneous ODE.

Problem 5. Consider the nonhomogeneous equation

$$y'' - 5y' + 6y = 6x + 1 + 4e^{4x}$$

(a) Verify that $\{e^{3x}, e^{2x}\}$ is a fundamental solution set for the associated homogeneous equation on $(-\infty, \infty)$.

(b) Verify that $y_{p_1} = x + 1$ is a solution of the nonhomogeneous equation $y'' - 5y' + 6y = 6x + 1$.

(c) Verify that $y_{p_2} = 2e^{4x}$ is a solution of the nonhomogeneous equation $y'' - 5y' + 6y = 4e^{4x}$.

(d) Write the general solution to the nonhomogeneous equation $y'' - 5y' + 6y = 6x + 1 + 4e^{4x}$.

(e) Sove the initial value problem

$$y'' - 5y' + 6y = 6x + 1 + 4e^{4x}, \quad y(0) = 0, \quad y'(0) = 6$$

Problem 6. Consider the nonhomogeneous equation

$$x^2y'' + xy' + y = 2x + \ln x$$

(a) Verify that $\{\cos(\ln x), \sin(\ln x)\}$ is a fundamental solution set for the associated homogeneous equation on $(0, \infty)$.

(b) Verify that $y_{p_1} = x$ is a solution of the nonhomogeneous equation $x^2y'' + xy' + y = 2x$.

(c) Verify that $y_{p_2} = \ln x$ is a solution of the nonhomogeneous equation $x^2y'' + xy' + y = \ln x$.

(d) Write the general solution to the nonhomogeneous equation $x^2y'' + xy' + y = 2x + \ln x$.

(e) Sove the initial value problem

$$x^2y'' + xy' + y = 2x + \ln x, \quad y(1) = 2, \quad y'(1) = 0$$

Problem 7. (Challenge Problem) Let P and Q be continuous on an interval I and consider the second order, linear homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0, \quad \text{for } x \text{ in } I. \tag{1.2}$$

(a) Suppose y_1 and y_2 are solutions of (1.2). Show that their Wronskian, $W(y_1, y_2)(x)$, satisfies the first order equation

$$\frac{dW}{dx} + P(x)W = 0. \tag{1.3}$$

(b) Show that for any x_0 in I

$$W(y_1, y_2)(x) = C_0 e^{-\int_{x_0}^{x} P(t)\,dt} \tag{1.4}$$

is the solution to (1.3) subject to the condition $W(y_1, y_2)(x_0) = C_0$.

Remark: The identity $W(y_1, y_2)(x) = W(y_1, y_2)(x_0)e^{-\int_{x_0}^{x} P(t)\,dt}$ is known as *Abel's identity*[2]. The formula that will be featured in the next section on *reduction of order* can be derived from Abel's identity.

(c) Note that the equation (1.4) shows that if y_1 and y_2 are solutions of (1.2), then the Wronskian is either identically zero or never zero on I. Use this to show that $y_1 = x^2$ and $y_2 = x^3 + 1$ cannot both be solutions of (1.2) on the interval $(-2, 2)$.

1.7 Section 7: Reduction of Order

Go to solutions 2.7

Get Acquainted Problem 7. Consider the second order equation

$$x^2y'' - 6xy' + 12y = 0 \quad \text{for} \quad x > 0. \tag{1.5}$$

(a) Verify that $y_1 = x^3$ is a solution of (1.5) on $(0, \infty)$.

[2] Abel's identity is one of several mathematical results named after Norwegian mathematician Niels Henrik Abel. He died at the young age of only 26 having made contributions to numerous fields of mathematics.

(b) Set $y_2 = x^3 u(x)$ where u is assumed to be at least two times differentiable. Find $\dfrac{dy_2}{dx}$ and $\dfrac{d^2 y_2}{dx^2}$.

(c) Substitute y_2 and its derivatives into the differential equation (1.5), and simplify to the extent possible. (You should obtain a second order equation for u that may have derivatives of u but no undifferentiated u terms.)

(d) Solve the resulting equation for $u(x)$. (This should be straight forward if your work is correct thus far.)

(e) With $y_2 = x^3 u(x)$, verify that y_2 is a solution of the differential equation (1.5) on $(0, \infty)$, and show that y_1 and y_2 are linearly independent.

(f) Write the general solution of the ODE (1.5). How many arbitrary constants do you have?

Problem 1. List the conditions under which Reduction of Order can be used to find the solution of a second order differential equation.

Problem 2. Consider the second order equation $y'' - 16y = 0$.

(a) Verify that $y_1 = e^{-4x}$ is a solution on $(-\infty, \infty)$.

(b) Set $y_2 = u(x)e^{-4x}$ where u is assumed to be at least two times differentiable. Find $\dfrac{dy_2}{dx}$ and $\dfrac{d^2 y_2}{dx^2}$.

(c) Substitute y_2 and its derivatives into the differential equation, and simplify to the extent possible. (You should obtain a second order equation for u that may have derivatives of u but no undifferentiated u terms.)

(d) Set $w = u'$. Your equation is a first order linear (and separable) equation for w. Solve this equation.

(e) Now find $u = \int w\, dx$. Set $y_2 = u(x)e^{-4x}$, verify that y_2 is a solution of the differential equation on $(-\infty, \infty)$, and show that y_1 and y_2 are linearly independent.

(f) Write the general solution of the ODE. How many arbitrary constants do you have?

Problem 3. Consider the differential equation $y'' - 4y' + 3y = 0$ which is in the form $y'' + P(x)y' + Q(x)y = 0$.

(a) One solution is $y_1 = e^x$. Use the formula we derived in class to obtain a second linearly independent solution. That is, compute

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

(b) A second solution to the equation is $y_2 = e^{3x}$. How does this solution differ from the one you just found? Why are both of these solutions equally valid?

Problem 4. Use the formula derived in class (shown in problem 3 above) to find a second linearly independent solution to each equation for which one solution has been provided. Write the general solution to each ODE.

(a) $y'' - 5y' + 4y = 0, \quad y_1 = e^x$

(b) $xy'' - (x+1)y' + y = 0, \quad y_1 = e^x$

(c) $x^2y'' + xy' + y = 0, \quad y_1 = \cos(\ln x)$

(d) $x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2$

Problem 5. Let h be any nonzero constant.

(a) Show that $f_1(x) = e^{hx}$ and $f_2(x) = xe^{hx}$ are linearly independent on $(-\infty, \infty)$.

(b) Show that $y_1 = e^{hx}$ is a solution of the ODE $y'' - 2hy' + h^2y = 0$ on $(-\infty, \infty)$.

(c) Use reduction of order to find a second linearly independent solution.

(d) Solve the initial value problem

$$y'' + 8y' + 16y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

Problem 6. Let k be any nonzero constant.

(a) Show that the functions $f_1(x) = x^k$ and $f_2(x) = x^k \ln x$ are linearly independent on $(0, \infty)$.

(b) Show that $y_1 = x^k$ is a solution to the ODE $x^2y'' - (2k-1)xy' + k^2y = 0$ on $(0, \infty)$.

(c) Use reduction of order to find a second linearly independent solution.

(d) Solve the initial value problem

$$x^2y'' - 5xy' + 9y = 0, \quad y(1) = 2, \quad y'(1) = -2.$$

Problem 7. (Adapted from [1]) The equation $(1 - x^2)y'' - xy' + \alpha^2 y = 0$ where α is a real[3] parameter is called a Chebyshev equation. Verify that for $\alpha = 1$, one solution of this equation is $y_1 = x$. Find a second linearly independent solution. Use your findings to solve the initial value problem

$$(1 - x^2)y'' - xy' + y = 0 \quad y(0) = 2 \quad y'(0) = -1$$

1.8 Section 8: Homogeneous Equations with Constant Coefficients

Write the definition of each term (phrase).

Glossary Item 31. Linear, homogeneous, constant coefficient ODE

Glossary Item 32. Characteristic polynomial

Glossary Item 33. Characteristic (or axiliary) equation

Go to solutions 2.8

Get Acquainted Problem 8. Consider the differential equation $y'' - 7y' + 10y = 0$.

(a) Let $y = e^{mx}$ for constant m. Find the first two derivatives of y.

(b) Substitute the derivatives you found into the ODE, and factor out the common e^{mx}.

(c) Divide by the exponential, and solve the resulting quadratic equation for solutions m_1 and m_2.

(d) Using the numbers m_1, m_2 you just found, verify that $y = c_1e^{m_1x} + c_2e^{m_2x}$ solves the ODE for any choice of the parameters c_1, c_2.

[3] In general we can allow α to be complex.

Problem 1. For each homogeneous equation, identify the characteristic polynomial **if the equation has one**. If an equation does not have a characteristic polynomial, state why.

(a) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - 3y = 0$

(b) $y^{(6)} + 4y^{(5)} - y''' + 6y'' - y' + 2y = 0$

(c) $xy'' + 2y' + x^2y = 0$

(d) $\dfrac{d^3x}{dt^3} - 12\dfrac{d^2x}{dt^2} + 48\dfrac{dx}{dt} - 64x = 0$

(e) $y^2\dfrac{dy}{dx} + x^2y = 0$

Problem 2. Let h be any nonzero real number, and consider the second order equation

$$\frac{d^2y}{dx^2} - 2h\frac{dy}{dx} + h^2y = 0$$

(a) Write out and solve the characteristic equation. (Use m as the parameter.)

(b) Explain in your own words why neither

$$y = c_1e^{hx}, \quad \text{nor} \quad y = c_1e^{hx} + c_2e^{hx}$$

is the **general solution**.

(c) Consider the case that $h = 1$. Show that it is not possible to find values of c_1 and c_2 such that the function $y = c_1e^x + c_2e^x$ satisfies the pair of conditions $y(0) = 1$, $y'(0) = 2$.

Problem 3. Find the general solution to each second order equation.

(a) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - 3y = 0$

(b) $y'' + 4y' + y = 0$

(c) $y'' + 4y' + 7y = 0$

(d) $9\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + x = 0$

(e) $\dfrac{d^2q}{dt^2} + 20\dfrac{dq}{dt} + 500q = 0$

(f) $x'' + \omega^2 x = 0$, where ω is a nonzero, real number

(g) $x'' - \lambda^2 x = 0$, where λ is a nonzero, real number

Problem 4. Solve each initial value problem.

(a) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - 3y = 0, \quad y(0) = 0, \quad y'(0) = -1$

(b) $y'' + 4y' + 8y = 0, \quad y(0) = 3, \quad y'(0) = 2$

(c) $y'' - 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2$

(d) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - y = 0, \quad y(0) = 2, \quad y'(0) = 0$

(e) $x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$ where ω is a nonzero, real number

(f) $x'' - \lambda^2 x = 0, \quad x(0) = 0, \quad x'(0) = 1$, where λ is a nonzero, real number

Problem 5. Find the general solution of each homogeneous equation.

(a) $\dfrac{d^3x}{dt^3} - 12\dfrac{d^2x}{dt^2} + 48\dfrac{dx}{dt} - 64x = 0$ (Hint: The characteristic polynomial is a perfect cube.)

(b) $y''' - y'' + 4y' - 4y = 0$ (Hint: The characteristic polynomial is easily factored by grouping.)

(c) $\dfrac{d^4y}{dx^4} - 81y = 0$ (Hint: Start with difference of squares.)

(d) $y^{(4)} - y^{(3)} - 6y^{(2)} = 0$

Problem 6. Consider the initial value problem

$$y'' - 4y' + 3y = 30e^{-2x}, \quad y(0) = 3, \quad y'(0) = 3$$

(a) Find the complementary solution y_c.

(b) Verify that $y_p = 2e^{-2x}$ is a particular solution.

(c) What is the general solution of the nonhomogeneous equation?

(d) Solve the IVP.

Problem 7. Consider the initial value problem

$$y''' - y'' + 4y' - 4y = 8x, \quad y(0) = -1, \quad y'(0) = 1, \quad y''(0) = 6$$

(a) Find the complementary solution y_c.

(b) Verify that $y_p = -2x - 2$ is a particular solution.

(c) What is the general solution of the nonhomogeneous equation?

(d) Solve the IVP.

Problem 8. The equation $x^2 \dfrac{d^2y}{dx^2} + ax\dfrac{dy}{dx} + by = 0$ for $x > 0$ is called a *Cauchy-Euler*[4] equation.

(a) Look for solutions in the form $y = x^r$. Show that such a power function solves the ODE provided r satisfies a certain quadratic equation.

(b) Find two functions $y = x^{r_1}$ and $y = x^{r_2}$ that solve the ODE

$$x^2y'' + 2xy' - 6y = 0.$$

Write the general solution as $y = c_1x^{r_1} + c_2x^{r_2}$.

(c) Use your findings to solve the IVP

$$x^2y'' + 2xy' - 6y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

Problem 9. **(Challenge Problem)** Consider the second order Cauchy-Euler equation $x^2 \dfrac{d^2y}{dx^2} + ax\dfrac{dy}{dx} + by = 0$ for $x > 0$. Introduce a new variable z by letting $x = e^z$. Let the new dependent variable u be defined by

$$u(z) = y(x).$$

[4]It's also called *equidimensional*, notice that the order of the derivative matches the degree of the monomial coefficient for each term.

(a) Use the chain rule to show that

$$\frac{du}{dz} = x\frac{dy}{dx}.$$

Hint: The chain rule tells us that $\frac{d}{dz}y = \frac{dy}{dx}\frac{dx}{dz}$.

(b) Use the product rule along with the chain rule to show that

$$\frac{d^2u}{dz^2} = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}.$$

Hint 1: According to the chain rule, the operation $\frac{d}{dz} = \left(\frac{dx}{dz}\right)\frac{d}{dx}$, i.e. taking a derivative with respect to z corresponds to taking a derivative with respect to x and multiplying by the factor $\frac{dx}{dz}$.
Hint 2: $\frac{d^2u}{dz^2} = \frac{d}{dz}\left(x\frac{dy}{dx}\right)$ requiring the product and chain rule.

(c) Using these results, show that u satisfies a constant coefficient equation. What is that equation?

(d) Since the Cauchy-Euler equation can be transformed into a constant coefficient equation, we have a general method for solving any Cauchy-Euler equation. Use this observation to find the general solution to each Cauchy-Euler equation. (Look for solutions of the form $y = x^r$ and use the connection to constant coefficient equations to deal with the repeated root and complex root cases.)

(i) $x^2y'' - 5xy' + 8y = 0$

(ii) $x^2y'' - 3xy' + 4y = 0$ (Hint: If $x = e^z$, then a factor $z = \ln x$.)

(iii) $x^2y'' + xy' + 9y = 0$ (Hint: If $x = e^z$, then a factor $z = \ln x$.)

1.9 Section 9: Method of Undetermined Coefficients

Go to solutions 2.9

Get Acquainted Problem 9. Consider the nonhomogeneous equation

$$y'' + 2y' + y = 3e^{6x}.$$

(a) Verify that $y_c = c_1e^{-x} + c_2xe^{-x}$ solves the associated homogeneous equation $y'' + 2y' + y = 0$.

(b) Set $y_p = Ae^{6x}$, where A is some constant. Take the derivatives and substitute this into the nonhomogeneous equation. Determine the value for A for which y_p solves the nonhomogeneous equation.

(c) Verify that $y = y_c + y_p$ solves the nonhomogeneous equation (using y_p that you determined).

Problem 1. The **method of undetermined coefficients** is only applicable when a linear, nonhomogeneous equation satisfies two conditions. One condition is on the left side of the equation, and the other on the right side of the equation. Suppose our generic linear, nonhomogeneous equation looks like

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

(a) What must be true of the left hand side for the method to apply?

(b) What must be true of the right side for the method to apply?

Problem 2. Find a particular solution to the equation that has the form provided.

(a) $y'' + 2y' + y = 4xe^{-x} \quad y_p = (Ax^3 + Bx^2)e^{-x}$

(b) $y'' + 2y' + y = 3\sin(2x) \quad y_p = A\sin(2x) + B\cos(2x)$

(c) $y'' + 16y = x^2 \quad y_p = Ax^2 + Bx + C$

(d) $y'' + 16y = e^{2x}\cos(3x) \quad y_p = Ae^{2x}\cos(3x) + Be^{2x}\sin(3x)$

(e) $y''' - y'' + y' - y = x^2 \quad y_p = Ax^2 + Bx + C$

Problem 3. Consider the nonhomogeneous equation $y'' + 2y' - 8y = 3x$.

(a) Try to find a particular solution of the form $y_p = Ax$. Why does this fail?

(b) What should the form of y_p be? Find a particular solution of this correct form.

Problem 4. Find the general solution of each nonhomogeneous equation.

(a) $y'' + 2y' + y = x$

(b) $y'' - 2y' + 5y = -3e^{-2x}$

(c) $y''' - y'' + y' - y = e^{3x}$ (Hint: One of the roots of the characteristic equation is 1.)

(d) $y'' - 5y' + 6y = e^{2x}$

(e) $y'' - 4y' + 4y = 4e^{2x}$

(f) $y'' + 9y = x^2 - x$

(g) $y'' - 2y' + y = -50\cos(2x)$

(h) $y'' + 3y' = 17e^{x}\sin(x)$

(i) $y'' + 3y' = 16xe^{-x}$

Problem 5. Find the general solution of each nonhomogeneous equation. Use the principle of superposition to make the task of finding y_p more manageable. (The left hand sides match the corresponding equations in problem 4 above.)

(a) $y'' + 2y' + y = 3 - \sin x$

(b) $y'' - 2y' + 5y = 50x^2 - 5e^{3x}$

(c) $y''' - y'' + y' - y = 2x - 8\cos(x)$

(d) $y'' - 5y' + 6y = e^{x} + 4e^{4x} - 12e^{6x}$

Problem 6. Let ω and γ be positive constants, and consider the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0\cos(\gamma t) \quad \text{where } F_0 \text{ is constant.} \tag{1.6}$$

(a) Find the complementary solution.

(b) Suppose $\gamma \neq \omega$ (these are different constants). What is the form of the particular solution x_p?

(c) Suppose $\gamma = \omega$ (these are the same constant). What is the form of the particular solution x_p?

(d) Suppose $\omega = 2$ and $\gamma = 3$. Determine the general solution.

(e) Suppose $\omega = \gamma = 2$. Determine the general solution.

The ODE (1.6) is a model of an undamped harmonic oscillator subject to an oscillating driving force. The case $\omega = \gamma$ is called *pure resonance*. In this case, the mechanical system's natural frequency ω matches the external imposed frequency γ. The result is oscillatory motion with an unbounded amplitude. While pure resonance never really occurs in nature (there's always some damping in the physical world), even near resonance, $\omega \approx \gamma$, can have significant implications for a physical system causing potentially damaging vibrations.

Problem 7. Determine the form of the particular solution when using the method of undetermined coefficients. Do not find any of the coefficients A, B, etc.

(a) $y'' - 2y' + 5y = 2e^x + 4\cos(2x) - 3e^x\cos(2x)$

(b) $y'' + 2y' + y = x^2 + x^2e^{-x}$

(c) $y''' + 6y'' + 8y' = 4x - e^{-2x} + 5e^{2x}$

(d) $y^{(4)} + 2y'' + y = \sin x + \cos(2x)$

1.10 Section 10: Variation of Parameters

Write the definition of each term (phrase).

Glossary Item 34.

Go to solutions 2.10.

Get Acquainted Problem 10. Consider the nonhomogeneous equation

$$y'' + 2y' + y = e^{-x} \qquad -\infty < x < \infty$$

(a) Find a fundamental solution set $\{y_1, y_2\}$ using the techniques from section 1.8.

(b) Find the Wronskian $W(y_1, y_2)(x)$. Call it W for short.

(c) Let $g(x) = e^{-x}$, the right side of the ODE which is in standard form. Find the following two functions by evaluating the integrals (you can take the constants of integration to be zero).

$$u_1(x) = \int \frac{-g(x)y_2(x)}{W}\,dx \quad \text{and} \quad u_2(x) = \int \frac{g(x)y_1(x)}{W}\,dx$$

(d) Set $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$; simplify as much as possible. Verify that this is a particular solution to the differential equation.

(e) Solve the initial value problem $y'' + 2y' + y = e^{-x}, \quad y(0) = 1, \quad y'(0) = 1$

Problem 1. Use variation of parameters to find a particular solution to each nonhomogeneous equation. If the complementary solution is not provided, it is assumed that you can find it using the techniques developed in this class.

(a) $y'' + 4y = \tan(2x), \quad -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$

(b) $y'' + 4y = \sec(2x), \quad -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$

(c) $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

(d) $y'' - 4y' + 4y = \sqrt{x}e^{2x}, \quad x > 0$

(e) $x^2y'' + xy' - y = x^3, \quad x > 0 \qquad y_c = c_1x + \dfrac{c_2}{x}$

(f) $x^2y'' + xy' - y = \dfrac{8\ln x}{x}, \quad x > 0 \qquad y_c = c_1x + \dfrac{c_2}{x}$

(g) $x^2y'' - 4xy' + 6y = x^{5/2}, \quad x > 0 \qquad y_c = c_1x^2 + c_2x^3$

(h) $x^2y'' - 4xy' + 6y = x^3, \quad x > 0 \qquad y_c = c_1x^2 + c_2x^3$

(i) $x^2y'' - 5xy' + 9y = x^3 \quad x > 0 \qquad y_c = c_1x^3 + c_2x^3\ln x$

(j) $4xy'' + 2y' + y = \sin\sqrt{x}, \quad x > 0 \qquad y_c = c_1\cos\sqrt{x} + c_2\sin\sqrt{x}$

Problem 2. Consider the initial value problem

$$y'' + 2y' + y = 70x^{3/2}e^{-x} + x, \quad y(0) = 0, \quad y'(0) = 1$$

(a) Find the complementary solution.

(b) Find a particular solution y_{p_1} to the ODE $y''+2y'+y = 70x^{3/2}e^{-x}$ using Variation of Parameters.

(c) Find a particular solution y_{p_2} to the ODE $y'' + 2y' + y = x$ using the Method of Undetermined Coefficients.

(d) Find the solution to the IVP.

(e) Try setting up Variation of Parameters to find y_{p_2}. Can you see where use of the principle of superposition greatly simplifies the necessary computations?

Problem 3. Solve the initial value problem using any applicable approach.

$$y'' + y = 2 - \sec x, \quad y(0) = 1, \quad y'(0) = 1$$

1.11 Section 11: Linear Mechanical Equations

Write the definition of each term (phrase).

Glossary Item 35. Simple Harmonic Motion

Glossary Item 36. Displacement in equilibrium

Glossary Item 37. Free Damped Motion: Overdamped, Underdamped, and Critically damped

Glossary Item 38. Resonance frequency

Go to solutions 2.11

Get Acquainted Problem 11. A certain spring has a natural length of 1 ft with no mass attached. A 10 lb object is attached, and the new spring-mass length for the system is 18 inches.

(a) Determine the mass m in slugs, and the spring constant k in lb/ft using weight $W = mg$ and $W = k\delta x$. (Take $g = 32$ ft/sec^2, and identify δx in feet.)

(b) Determine the square of the natural (circular) frequency ω^2. Use both formulas to verify that they produce the same output ($\omega^2 = \dfrac{k}{m} = \dfrac{g}{\delta x}$)

(c) Assume that there is no damping, and no external driving force so that the displacement x satisfies $x'' + \omega^2 x = 0$. If the object starts 3 inches above equilibrium with an initial downward velocity of 2 ft/sec, determine the displacement for all $t > 0$.

(d) Determine the first time value at which the object reaches equilibrium.

(e) Find the amplitude of the motion. (Recall that if $y = a\cos(\theta) + b\sin(\theta)$, the amplitude $A = \sqrt{a^2 + b^2}$.)

Throughout these exercises, take the gravitational constant $g = 32$ ft/sec^2 or $g = 9.8$ m/sec^2 as appropriate.

Problem 1. An object stretches a spring 4 inches in equilibrium. Assume that there is no damping or external driving. If the object is initially displaced 1 foot below equilibrium with an initial upward velocity of 2 feet per second, determine the displacement for $t > 0$.

Problem 2. ([2]) A spring with natural length 0.5 m has length 50.5 cm with a mass of 2 gm suspended from it. The mass is initially displaced 1.5 cm below equilibrium and released with zero velocity. Find its displacement for $t > 0$.

Problem 3. To determine the spring constant of a certain spring, a 10 lb object is attached to it, and a 4 inch displacement is observed. The 10 lb object is removed. Later, a 2 lb object is attached to the spring and released from equilibrium with an initial upward velocity of 1 ft/sec. Assuming that there is no damping and no external driving, determine the displacement for all $t > 0$.

Problem 4. Consider the function $f(t) = a\cos(\omega t) + b\sin(\omega t)$ where a, b, and ω are real constants.

(a) Set $A = \sqrt{a^2 + b^2}$ and verify that $\left(\dfrac{a}{A}\right)^2 + \left(\dfrac{b}{A}\right)^2 = 1$.

Note that this tells us that the numbers $\frac{a}{A}$ and $\frac{b}{A}$ can be the sine and cosine of some angle.

(b) Define an angle ϕ by the pair of equations $\sin\phi = \frac{a}{A}$ and $\cos\phi = \frac{b}{A}$. Next, write $f(t) = A\left(\dfrac{1}{A}f(t)\right)$. Use the sum of angles formula for the sine function to express $f(t)$ as a single sine function.

(c) Alternatively, define an angle $\hat{\phi}$ by the pair of equations $\cos\hat{\phi} = \frac{a}{A}$ and $\sin\hat{\phi} = \frac{b}{A}$. Again writing $f(t) = A\left(\dfrac{1}{A}f(t)\right)$, use the difference of angles formula for the cosine function to express f as a single cosine term.

The advantage to writing such a form is that the amplitude A is explicit, and the period $T = \frac{2\pi}{\omega}$ is still easily identified. The value ϕ (or $\hat{\phi}$) is a phase shift.

Problem 5. (Adapted from [2]) An object stretches a spring 1.5 inches in equilibrium. The object is released from a position 8 inches above equilibrium with an initial downward velocity of 4 ft/sec. Assuming there is no damping and no driving, determine the displacement for all $t > 0$. Write the displacement $x(t)$ in the form $x(t) = A\sin(\omega t + \phi)$ and identify the amplitude, period, and phase shift of the motion.

Problem 6. ([2]) Two objects are attached to identical springs and set in motion (simple harmonic motion). The period of one objects is observed to be twice the period of the second object. How are the object's masses related?

Problem 7. A 64 lb object is attached to a spring whose spring constant is 26 lb/ft. A dashpot provides damping that is numerically equal to 8 times the instantaneous velocity.

(a) Determine the mass m of the object in slugs.

(b) Assuming there is no external applied force, set up the differential equation for the displacement and determine if the motion is overdamped, underdamped or critically damped.

(c) If the object is initially displaced 6 inches above equilibrium and given an initial upward velocity of 2 ft/sec, determine the displacement for $t > 0$.

Problem 8. A 1 kg mass is attached to a spring whose spring constant is 9 N/m. The surrounding medium offers a damping force that is 6 times the instantaneous velocity.

(a) In the absence of external driving, set up the differential equation for the displacement x, and determine if the system is overdamped, underdamped, or critically damped.

(b) Assuming there is no external driving, if the mass is released from a position 10 cm above equilibrium with an initial upward velocity of 0.5 m/sec, determine the displacement x for all $t > 0$.

(c) Suppose instead that a driving force $F(t) = \cos(t)$ is applied. Now if the mass is released from a position 10 cm above equilibrium with an initial upward velocity of 0.5 m/sec, determine the displacement x for all $t > 0$.

Problem 9. A 64 lb object stretches a spring 6 inches in equilibrium. The surrounding medium provides a damping force of b lbs for each ft/sec of velocity.

(a) Find the mass m in slugs and the spring constant k, and write out the equation $mx'' + bx' + kx = 0$.

(b) Taking into account that $b \geq 0$, determine the values of b for which the resulting system would be underdamped, overdamped, and critically damped.

Problem 10. A 192 lb object is attached to a spring with spring constant 150 lb/ft. There is no damping, but an external driving force of $f(t) = F_0 \cos(\gamma t)$ is applied.

(a) Determine the value of γ that would induce pure resonance.

(b) Suppose $F_0 = 6$ and $\gamma = 1$ per second (note this is not the resonance frequency you found above). If the object is released from rest at equilibrium (zero initial displacement and velocity), determine the displacement for all $t > 0$.

(c) Suppose $F_0 = 6$ and γ is the resonance frequency. If the object is released from rest at equilibrium, determine the displacement for all $t > 0$.

Problem 11. Solve the pure resonance initial value problem

$$x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

1.12 Section 12: LRC Series Circuits

Write the definition of each term (phrase).

Glossary Item 39. TRANSIENT STATE CHARGE, AND STEADY STATE CHARGE (of an LRC series circuit)

Go to solutions 2.12

Get Acquainted Problem 12. Suppose an LRC series circuit has constant inductance, resistance, and capacitance L_0, R_0 and C_0, respectively (in appropriate units). Further suppose that a constant electromotive force $E(t) = E_0$ volts is applied.

(a) Write out the second order, linear, nonhomogeneous ($E_0 \neq 0$) differential equation satisfied by the charge q on the capacitor.

(b) Use the fact that the current $i = \dfrac{dq}{dt}$, and take the derivative of the equation for the charge to obtain a second order, linear, homogeneous differential equation for the current i.

Problem 1. [2] Solve the equation describing free electrical vibrations for each set of parameters. Determine the charge q on the capacitor and the current i in the circuit.

(a) $L = 0.05$ h, $R = 2$ ohms, $C = 0.01$ f, $q_0 = 2$ c, and $i_0 = -2$ A

(b) $L = 0.1$ h, $R = 2$ ohms, $C = 0.01$ f, $q_0 = 2$ c, and $i_0 = 0$ A

(c) $L = 0.1$ h, $R = 6$ ohms, $C = 0.004$ f, $q_0 = 3$ c, and $i_0 = -10$ A

Problem 2. A dying battery imparts an electromotive force $E(t) = 10e^{-t}$ to an LC-series circuit (there is no resistor). The inductance is 1 henry and the capacitance is 0.01 (i.e. $\frac{1}{100}$) farads. If the initial charge on the capacitor $q(0) = 0$ and the initial current $i(0) = 0$, determine the charge $q(t)$ for all $t > 0$.

Problem 3. An LRC-series circuit has inductance 1 henry, resistance 2 ohms and capacitance 1 farad. A voltage of $E(t) = 8e^{-3t}$ is applied to the circuit. If the initial charge $q(0) = 0$ and the initial current $i(0) = 0$, determine the charge $q(t)$ on the capacitor for all $t > 0$.

Problem 4. [2] Find the steady state **current** in the LRC series circuit described by the equation

$$\frac{1}{10}q'' + 3q' + 100q = 5\cos(10t) - 5\sin(10t)$$

Introduction to Laplace Transforms

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s} \quad s > 0$
$t^n \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}} \quad s > 0$
$t^r \quad r > -1$	$\frac{\Gamma(r+1)}{s^{r+1}} \quad s > 0$
e^{at}	$\frac{1}{s-a} \quad s > a$
$\sin(kt) \quad k \neq 0$	$\frac{k}{s^2+k^2} \quad s > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2} \quad s > 0$
$e^{at} f(t)$	$F(s-a)$
$\mathscr{U}(t-a) \quad a > 0$	$\frac{e^{-as}}{s} \quad s > 0$
$\mathscr{U}(t-a) f(t-a) \quad a > 0$	$e^{-as} F(s)$
$\mathscr{U}(t-a) g(t) \quad a > 0$	$e^{-as} \mathscr{L}\{g(t+a)\}$
$\delta(t-a) \quad a \geq 0$	e^{-as}
$(f * g)(t)$	$F(s)G(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
$t f(t)$	$-\frac{d}{ds} F(s)$
$t^n f(t) \quad n = 1, 2, \ldots$	$(-1)^n \frac{d^n}{ds^n} F(s)$
$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$

Gamma Function:[5] $\Gamma(n+1) = n!$ **for** $n = 0, 1, 2, \ldots,$ **and for** $x > 0$**,**

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

Convolution: For f **and** g **piecewise continuous on** $[0, \infty)$**,**

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau.$$

[5]The entry for t^r reduces to that for t^n if r is a positive integer but is also valid for noninteger powers, e.g. $\sqrt{t}$.

1.13 Section 13: The Laplace Transform

Write the definition of each term (phrase).

Glossary Item 40. The Laplace transform

Go to solutions 2.13

Get Acquainted Problem 13. Suppose we wish to determine the Laplace transform of $y(t) = \cos(2t)\sin(2t) + 2t^2$ using a table of Laplace transforms.

(a) Use the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to write the product $\cos(2t)\sin(2t)$ as a single sine term.

(b) Use the table entry $\mathscr{L}\{\sin(kt)\} = \dfrac{k}{s^2+k^2}$ to find the transform of the sine term you just found.

(c) Use the table entry $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ to determine $\mathscr{L}\{t^2\}$.

(d) Using the linearity property $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathscr{L}\{f(t)\} + \beta\mathscr{L}\{g(t)\}$ from the table, and the transforms you've just computed, evaluate

$$\mathscr{L}\{y(t)\}.$$

Problem 1. Use the definition of the Laplace transform (i.e. compute an integral) to evaluate $F(s) = \mathscr{L}\{f(t)\}$ for the following functions.

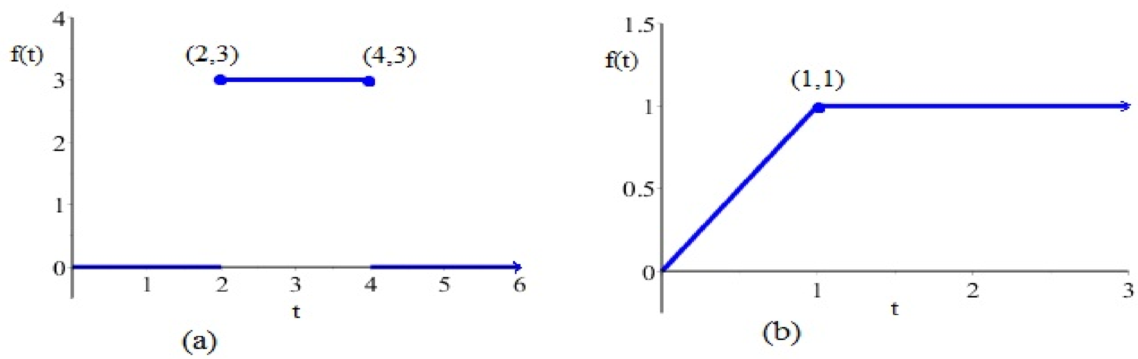
(a) $f(t) = e^{3t}$

(b) $f(t) = e^{t-2}$

(c) $f(t) = \begin{cases} 2, & 0 \le t < 2 \\ 1, & t \ge 2 \end{cases}$

(d) $f(t) = \begin{cases} e^t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

Problem 2. Determine the Laplace transform of each function given its graph.



Problem 3. Use the table of Laplace transforms to determine $F(s) = \mathscr{L}\{f(t)\}$ for the following functions. Make use of algebra or function identities as needed. (It is not necessary to indicate the domain of F.)

(a) $f(t) = 3t^4 - 2t^2 + t + 4$

(b) $f(t) = \cos(3t) + \dfrac{1}{2}\sin(3t)$

(c) $f(t) = at + b$ for a, b real numbers

(d) $f(t) = \dfrac{e^t + e^{-t}}{2}$

(e) $f(t) = (2t-1)^2$

(f) $f(t) = (1 - e^{-3t})^2$

(g) $f(t) = (e^t - e^{-t})^2$

(h) $f(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$, where α, β and ω are real numbers

(i) $f(t) = \sin\left(t - \frac{\pi}{6}\right)$

(j) $f(t) = \sin^2(\omega t)$ for real number ω

Problem 4. While the transform of the sum is the sum of the transforms,

$$\text{i.e.} \quad \mathscr{L}\{f+g\} = \mathscr{L}\{f\} + \mathscr{L}\{g\},$$

there is no analogous property for products. Unfortunately, people who are inexperienced often attempt to apply such a property. Convince yourself that seemingly *obvious* shortcuts are just wrong.

(a) Note that $t^2 = t \cdot t$. Show that $\mathscr{L}\{t^2\}$ is NOT equal to $\mathscr{L}\{t\} \cdot \mathscr{L}\{t\}$.

(b) Recall the trigonometric identity $\sin(2t) = 2 \sin t \cos t$. Show that $\mathscr{L}\{\sin(2t)\}$ is NOT equal to $2\mathscr{L}\{\sin t\} \cdot \mathscr{L}\{\cos t\}$.

(c) Use properties of exponentials to write the product $e^{at}e^{bt}$ as a single exponential.

(d) Show that $\mathscr{L}\{e^{at}e^{bt}\}$ is NOT equal to $\mathscr{L}\{e^{at}\} \cdot \mathscr{L}\{e^{bt}\}$.

The Laplace transform is an integral. It inherits properties of integrals. Remember that the integral of a product is NOT the product of the integrals. That is, $\int f(x)g(x)\,dx$ and $(\int f(x)\,dx)(\int g(x)\,dx)$ are very different animals!

Problem 5. **(Challenge Problem)** Suppose we wish to find $\mathscr{L}\{\sqrt{t}\}$. There is a table entry for non-integer powers. It depends on the Gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt, \qquad \text{for} \quad x > 0$$

(a) Assuming $r + 1 > 0$, write the integral for $\Gamma(r+1)$ from the definition of Γ. (This is just a straightforward substitution.)

(b) Using the definition of the Laplace transform, show that if $r > -1$

$$\mathscr{L}\{t^r\} = \frac{\Gamma(r+1)}{s^{r+1}} \quad \text{for} \quad s > 0$$

Hint: Use the change of variables $\tau = st$. Remember that the name of the dummy variable is immaterial.

(c) Use integration by parts to show that $\Gamma(r+1) = r\Gamma(r)$ for any $r > 0$
Hint: Recall that for any real number r, $e^{-t}t^r \to 0$ as $t \to \infty$.

(d) Use the your results, along with the known value $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ to find the Laplace transform of each function.

(i) $f(t) = t^{-1/2}$

(ii) $f(t) = \sqrt{t}$

(iii) $f(t) = t^{3/2}$

1.14 Section 14: Inverse Laplace Transforms

Write the definition of each term (phrase).

Glossary Item 41. An inverse Laplace transform

Go to solutions 2.14.

Get Acquainted Problem 14. Suppose you wish to compute an inverse Laplace transform

$$\mathscr{L}^{-1}\left\{\frac{1}{s^3}+\frac{1}{s^2-3s}\right\}$$

(a) First note that $\frac{1}{s^3}=\frac{1}{2!}\frac{2!}{s^3}$. Use the table entry $\mathscr{L}\{t^n\}=\frac{n!}{s^{n+1}}$ to find $\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\}$.

(b) Now, do a partial fraction decomposition on the ratio $\frac{1}{s^2-3s}=\frac{1}{s(s-3)}$.

(c) Use the table entries $\mathscr{L}\{1\}=\frac{1}{s}$ and $\mathscr{L}\{e^{at}\}=\frac{1}{s-a}$ to find $\mathscr{L}^{-1}\left\{\frac{1}{s^2-3s}\right\}$.

(d) Pull it all together and evaluate $\mathscr{L}^{-1}\left\{\frac{1}{s^3}+\frac{1}{s^2-3s}\right\}$.

Problem 1. Evaluate each inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s-2}\right\}$

(b) $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\}$

(c) $\mathscr{L}^{-1}\left\{\frac{3}{s}+\frac{4}{s^2}+\frac{5}{s^3}\right\}$

(d) $\mathscr{L}^{-1}\left\{\frac{1}{2s-6}\right\}$

Problem 2. Find a general result for $\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\}$ for $n\geq 2$.

Problem 3. Find a general result for $\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\}$ for $k>0$.

Problem 4. For each rational function, perform a partial fraction decomposition, and then obtain an inverse Laplace transform.

(a) $F(s) = \dfrac{1}{s^2+s-2}$

(b) $F(s) = \dfrac{s}{s^2+2s-3}$

(c) $F(s) = \dfrac{s-1}{s(s^2+1)}$

(d) $F(s) = \dfrac{2s}{(s^2-1)(s+2)}$

(e) $F(s) = \dfrac{s+1}{(s^2+1)(s^2+4)}$

(f) $F(s) = \dfrac{2s^3+s^2+9}{s^3(s^2+9)}$

Problem 5. The hyperbolic sine and cosine functions are sums of real exponentials. They are denoted by $\sinh(x)$ and $\cosh(x)$, respectively and are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{for all real } x.$$

Use the table entries for e^{at} to obtain a Laplace transform result for the hyperbolic sine and cosine. That is, for $k \neq 0$, find a (simplified) formula for

(a) $\mathscr{L}\{\sinh(kt)\}$

(b) $\mathscr{L}\{\cosh(kt)\}$

Problem 6. Convince yourself that there is no shortcut to finding an inverse Laplace transform of a product.

(a) Note that $\dfrac{1}{(s-1)(s+2)} = \left(\dfrac{1}{s-1}\right)\left(\dfrac{1}{s+2}\right)$.

Show that $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-1)(s+2)}\right\}$ is NOT equal to $\mathscr{L}^{-1}\left\{\dfrac{1}{s-1}\right\} \cdot \mathscr{L}^{-1}\left\{\dfrac{1}{s+2}\right\}$.

(b) Note that $\frac{1}{s^5} = \left(\frac{1}{s^2}\right)\left(\frac{1}{s^3}\right)$.

Show that $\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}$ is NOT equal to $\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} \cdot \mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\}$.

Doing a partial fraction decomposition may be time consuming, but in general there is no shortcut to getting a correct result.

Problem 7. There is a result for the inverse Laplace transform of a product in terms of a *convolution*.

Definition (convolution): Suppose f ang g are integrable on $[0, \infty)$, the convolution of f and g is denoted by $(f * g)(t)$ and is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau, \quad t \geq 0$$

Theorem: If $F(s) = \mathscr{L}\{f(t)\}$ and $G(s) = \mathscr{L}\{g(t)\}$, then $\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$.

(a) Compute $\mathscr{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}$ in two ways. One using a partial fraction decomposition, and again using convolution with $F(s) = \frac{1}{s-1}$ and $G(s) = \frac{1}{s+2}$. (You should have already completed the first half of this exercise when working the previous problem.)

(b) Use a convolution to evaluate $\mathscr{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

Problem 8. Find the Laplace transform or inverse Laplace transform using the convolution (see problem 7).

(a) $\mathscr{L}\left\{\int_0^t \cos(2\tau)\cos(4(t-\tau))\,d\tau\right\}$

(b) $\mathscr{L}^{-1}\left\{\frac{4}{s(s^2+16)}\right\}$

(c) $\mathscr{L}^{-1}\left\{\frac{1}{s^3(s-2)}\right\}$

(d) **(Challenge Problem)** $\mathscr{L}\left\{\frac{1}{(s+1)(s-1)(s-2)}\right\}$

Problem 9. The convolution has various properties. Show each of the following.

(a) Let $f(t) = t$ and $g(t) = e^{2t}$. Show that $f * g = g * f$.

(b) Show that $f * g = g * f$ for any pair of continuous functions f and g.

(c) Let $f(t) = t$. Show that $(f * 1)(t) \neq f(t)$.

(d) Can you find a general relationship between $(f * 1)(t)$ and $f(t)$ if f is some continuous function on $[0, \infty)$?

Problem 10. **(Challenge Problem)** Use the convolution and inverse Laplace transform to find the function f that satisfies

$$f(t) = \int_0^t f(\tau)e^{2(t-\tau)}\,d\tau + 3$$

1.15 Section 15: Shift Theorems

Write the definition of each term (phrase).

Glossary Item 42. Unit step function

Go to solutions 2.15

Get Acquainted Problem 15. This exercise has two parts, one for each shift theorem. You may wish to complete them in separate sittings.

(Part 1) Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2 - 4s + 8}\right\}$ by following a few steps.

(a) Note that the denominator of the rational expression is irreducible (doesn't factor). So first complete the square on $s^2 - 4s + 8$ to obtain an expression of the form $(s - a)^2 + k^2$.

(b) Now you have a rational expression $\dfrac{s}{(s - a)^2 + k^2}$. The shift in s theorem requires that you have a function $F(s - a)$. So use the fact that $s = s - a + a$, to rewrite the numerator and to write the rational expression as the sum of two rational expressions.

(c) Finally, use the shift theorem and other appropriate table entries to evaluate the inverse Laplace transform.

(Part 2) Evaluate $\mathscr{L}\{f(t)\}$ where $f(t) = \begin{cases} t, & 0 \le t < 2 \\ e^t, & t \ge 2 \end{cases}$ by following a few steps.

(a) Start by writing f as a sum (or difference) of expressions with step function factors as appropriate.

(b) For $g_1(t) = t$, determine what $g(t+a)$ is, and for $g_2(t) = e^t$, determine what $g_2(t+a)$ is where a is some fixed, positive number.

(c) Now, use the table entries, including $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$, to find the Laplace transform of f.

Problem 1. Evaluate $F(s) = \mathscr{L}\{f(t)\}$ for each function.

(a) $f(t) = e^{-t}\sin(3t)$

(b) $f(t) = 2te^{4t}$

(c) $f(t) = (t^2 + e^t)^2$

(d) $f(t) = e^{at}(At^3 + Bt^2 + Ct + D)$ for nonzero constants a, A, B, C, and D

(e) $f(t) = Ae^{-\lambda t}\cos(\omega t) + Be^{-\lambda t}\sin(\omega t)$, for nonzero constants A, B, λ and ω

Problem 2. Evaluate $f(t) = \mathscr{L}^{-1}\{F(s)\}$ for each function.

(a) $F(s) = \dfrac{1}{(s-3)^6}$

(b) $F(s) = \dfrac{1}{(s-1)^2(s+2)}$

(c) $F(s) = \dfrac{s}{s^2+6s+14}$

(d) $F(s) = \dfrac{s^2+s-1}{(s+2)^3}$

(e) $F(s) = \dfrac{25s}{(s+6)^2(s^2+9)}$

Problem 3. Evaluate $F(s) = \mathscr{L}\{f(t)\}$ for each function.

(a) $f(t) = t^2\mathscr{U}(t-3)$

(b) $f(t) = (1+e^{2t})\mathscr{U}(t-1)$

(c) $f(t) = \sin(\pi t)\mathscr{U}\left(t - \dfrac{1}{3}\right)$

(d) $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ e^{-t}\cos(2t), & t \ge \pi \end{cases}$

(e) $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ t^2, & t \ge 2 \end{cases}$

Problem 4. Evaluate $f(t) = \mathscr{L}^{-1}\{F(s)\}$ for each function.

(a) $F(s) = \dfrac{e^{-\frac{1}{2}s}}{s^2+25}$

(b) $F(s) = \dfrac{e^{-2s}}{s(s+4)}$

(c) $F(s) = \dfrac{e^{-s}}{s(s^2+1)}$

(d) $F(s) = \dfrac{e^{-2s}}{(s-5)^4}$

Problem 5. Sometimes there is more than one approach to using the table entries to compute a Laplace transform or inverse transform. Fortunately, any legitimate approach will lead to the same result. Suppose α is any nonzero number and $\beta > 0$. Compute

$$\mathscr{L}\{e^{\alpha t}\mathscr{U}(t-\beta)\}$$

in two ways.

(a) Use $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$ with $F(s) = \mathscr{L}\{\mathscr{U}(t-\beta)\}$.

(b) Use $\mathscr{L}\{f(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}$ with $f(t) = e^{\alpha t}$.

Problem 6. (Challenge Problem) Use the definition of the Laplace transform (as an integral) to show that if f has a Laplace transform, then for any $a > 0$

(a) $\mathscr{L}\left\{f(t-a)\mathscr{U}(t-a)\right\} = e^{-as}\mathscr{L}\{f(t)\}$, and

(b) $\mathscr{L}\left\{f(t)\mathscr{U}(t-a)\right\} = e^{-as}\mathscr{L}\{f(t+a)\}$.

(Hint: Use a suitable u-substitution.)

1.16 Section 16: Laplace Transforms of Derivatives and IVPs

Go to solutions 2.16

Get Acquainted Problem 16. Consider the first order linear initial value problem

$$\frac{dy}{dt} + 3y = t, \quad y(0) = 1.$$

Take the following steps to solve this IVP using the Laplace transform. Let $Y(s)$ denote $\mathscr{L}\{y(t)\}$.

(a) Take the Laplace transform of both sides of the ODE making use of the result $\mathscr{L}\{y'(t)\} = s\mathscr{L}\{y(t)\} - y(0)$.

(b) Substitute in the known initial value $y(0) = 1$, and use any necessary algebra to isolate the term $Y(s)$ in your equation.

(c) Using partial fraction decomposition and any other necessary algebra, take the inverse Laplace transform to obtain $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

(d) Verify that the inverse transform $y(t)$ that you found satisfies both the ODE and the initial condition.

(e) Solve the equation again using an integrating factor (in the spirit of section 1.4), and finding the constant by applying the initial condition at the end. **(This last step is not part of the solution process, it is only included to reinforce that this new technique is an alternative approach to solving the IVP.)**

Problem 1. Solve each initial value problem using the method of Laplace transforms.

(a) $y'' + 3y' + 2y = e^t, \quad y(0) = 1, \quad y'(0) = -6$ [2]

(b) $y'' + y = \sin(2t), \quad y(0) = 0, \quad y'(0) = 1$ [2]

(c) $y'' + y' - 2y = -4, \quad y(0) = 2, \quad y'(0) = 3$ [2]

(d) $y'' + y' = 2e^{3t}, \quad y(0) = -1, \quad y'(0) = 4$ [2]

(e) $y''' = 1, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 2$

Problem 2. Solve each initial value problem using the method of Laplace transforms.

(a) $y'' - 6y' + 9y = e^{3t}, \quad y(0) = 1, \quad y'(0) = 0$

(b) $y'' + 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = -1$

(c) $y'' + 8y' + 16y = t^4 e^{-4t}, \quad y(0) = 1, \quad y'(0) = 1$

Problem 3. Solve each initial value problem.

(a) $y'' + y = \mathscr{U}(t-1), \quad y(0) = 0, \quad y'(0) = 0$

(b) $y'' - y = t\mathscr{U}(t-1), \quad y(0) = 0, \quad y'(0) = 1$

Problem 4. Solve the initial value problem using the method of Laplace transforms.

(a) $\dfrac{dy}{dt} + y = \begin{cases} 1, & 0 \le t < 3 \\ 0, & t \ge 3 \end{cases} \quad y(0) = 0$ (This problem appeared in section 1.4.)

(b) $y'' + y = \begin{cases} 3, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases} \quad y(0) = 0, \quad y'(0) = 0$ [2]

Problem 5. A 1 kg mass is attached to a spring with spring constant 16 N/m. No damping is applied, but a 16 N external stretching force is applied for 2 seconds and then released. If the mass starts from rest at equilibrium, determine the displacement of the mass for all time $t > 0$. That is, solve the following initial value problem using the method of Laplace transforms.

$$x'' + 16x = 16 - 16\mathscr{U}(t-2), \qquad x(0) = 0, \quad x'(0) = 0$$

Problem 6. An RC-series circuit has resistance 5 ohms and capacitance 0.01 farads. Initially, the charge on the capacitor $q(0) = 0$ coulombs. At time $t = 1$ seconds, a unit impulse voltage is applied. That is, the implied electromotive force for the system is $E(t) = \delta(t-1)$. It is known that $\mathscr{L}\{\delta(t-a)\} = e^{-as}$. Determine the charge $q(t)$ on the capacitor for all $t > 0$. Note that q satisfies the IVP

$$5\frac{dq}{dt} + 100q = \delta(t-1) \qquad q(0) = 0$$

Problem 7. Solve the system of equations using the method of Laplace transforms. (If you think carefully on this, you can probably guess the solution. But it's good to go through the steps to practice the process.)

$$\begin{aligned} \frac{dx}{dt} &= -y, \quad & x(0) = 1 \\ \frac{dy}{dt} &= x, \quad & y(0) = 0 \end{aligned}$$

Problem 8. Solve the system of equations using the method of Laplace transforms.

$$\begin{aligned}\frac{dx}{dt} &= x + 3y, \quad x(0) = 4 \\ \frac{dy}{dt} &= x - y, \quad y(0) = 0\end{aligned}$$

Problem 9. Solve the system of equations using the method of Laplace transforms.

$$\begin{array}{ccccccc} x''(t) & + & x(t) & - & 3y(t) & = & 0 \\ y''(t) & + & 3y(t) & - & x(t) & = & 0 \end{array}$$

subject to the initial conditions

$$x(0) = 8 \qquad x'(0) = 0$$

$$y(0) = 0 \qquad y'(0) = 4$$

Problem 10. Solve the pure resonance initial value problem, where F_0 is a nonzero real number and $\omega > 0$. (The convolution can be used to take the inverse transform.)

$$x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

An Introduction to Fourier Series

1.17 Section 17: Fourier Series: Trigonometric Series

Write the definition of each term (phrase).

Glossary Item 43. FOURIER SERIES OF A FUNCTION f ON $(-\pi, \pi)$ (Trigonometric series)

Glossary Item 44. FOURIER SERIES OF A FUNCTION f ON $(-p, p)$ (Trigonometric series)

Glossary Item 45. CONVERGENCE IN THE MEAN (for a Fourier series)

Go to solutions 2.17

Get Acquainted Problem 17. Consider the function $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$ Construct the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{1.7}$$

by following the given steps. Throughout, it is useful to recall that

$$\sin(n\pi) = 0 \quad \text{and} \quad \cos(n\pi) = (-1)^n \quad \text{for every integer } n.$$

(a) Find the coefficient a_0 using the integral formula $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$.

(b) Find the coefficients a_n using the integral formula $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$.

(c) Find the coefficients b_n using the integral formula $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$.

(d) Substitute these into the general format shown in equation (1.7). Remember to divide the value a_0 you found by two.

Problem 1. Suppose f is integrable on the interval $[a, b]$.

(a) What is the average value of f on $[a, b]$? (If you don't recall, any calculus text or a quick search will provide the answer.)

(b) Show that if f is integrable on $[-\pi, \pi]$, the value $\frac{a_0}{2}$ in the Fourier series of f is the average value of f on $[-\pi, \pi]$.

There is nothing special about $p = \pi$. The constant term in the Fourier series for a function is its average value on the interval $(-p, p)$. If we consider a partial sum of a Fourier series as an approximation to the function from which it is generated, the first approximation to the function on the interval would be this average value.

Problem 2. Determine the Fourier series for the given function on $(-\pi, \pi)$.

(a) $f(x) = 1 + x, \quad -\pi < x < \pi$

(b) $f(x) = x^2, \quad -\pi < x < \pi$

(c) $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ x, & 0 \leq x < \pi \end{cases}$

Problem 3. Determine the Fourier series for the given function on $(-p, p)$.

(a) $f(x) = 2 - x, \quad -1 < x < 1$ [2]

(b) $f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1, & -1 \leq x \leq 1 \\ 0, & 1 < x < 2 \end{cases}$

(c) $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 2x, & 0 \leq x < 2 \end{cases}$

Problem 4. Show that

(a) $\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0$ for every pair of integers m and n

(b) $\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

(c) $\displaystyle\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

Problem 5. Consider the function $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 2x, & 0 \le x < 2 \end{cases}$ whose Fourier series you determined in the previous problem. Produce a plot of the Fourier series (what the series converges to) over the interval $[-6, 6]$. Use your knowledge of **convergence in the mean** to identify the values to which the series converges when $x = -2$, $x = -1$, $x = 0$, $x = 1$, $x = 2$, and $x = 4.5$

Problem 6. (Challenge Problem) In problem 2, you found the Fourier series for $f(x) = x^2$ on $(-\pi, \pi)$. Convergence of this 2π-periodic series can be used to obtain the value of the common p-series with $p = 2$ as well as its alternating version.

(a) Evaluate the function and its series at $x = 0$ to show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

(b) Use the series with a carefully chosen x value to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

1.18 Section 18: Sine and Cosine Series

Write the definition of each term (phrase).

Glossary Item 46. Even function

Glossary Item 47. Odd function

Glossary Item 48. Half range sine (cosine) series

Go to solutions 2.18

Get Acquainted Problem 18. Consider the function $f(x) = x,\ 0 < x < 1$. The following useful integrals are readily verified:

$$\int_0^1 x\,dx = \frac{1}{2} \quad \int_0^1 x\cos(n\pi x)\,dx = \frac{(-1)^n - 1}{n^2\pi^2}, \quad \text{and} \quad \int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^{n+1}}{n\pi}$$

(a) Write the half range sine series for f. (Use the integrals provided.)

(b) Write the half range cosine series for f. (Use the integrals provided.)

(c) Sketch a graph of the half range sine series for f over the interval $[-3,3]$. (Your plot should highlight the odd extension, the 2-periodicity, and convergence in the mean.)

(d) Sketch a graph of the half range cosine series for f over the interval $[-3,3]$. (Your plot should highlight the even extension, the 2-periodicity, and convergence in the mean.)

Problem 1. Suppose f and g are defined on $(-a,a)$ for some $a>0$. Show that

(a) If f and g are even, then fg is even.

(b) If f is even and g is odd, then fg is odd.

(c) If f is odd and g is odd, then fg is even.

Problem 2. Each of the following functions has either even or odd symmetry. Find the Fourier series of each function.

(a) $f(x)=|x|, \quad -\pi<x<\pi$

(b) $f(x)=\begin{cases} 1, & -1<x<-\frac{1}{2} \\ 0, & -\frac{1}{2}<x<\frac{1}{2} \\ -1, & \frac{1}{2}<x<1 \end{cases}$

(c) $f(x)=\begin{cases} 0, & -2<x<-1 \\ 1+x, & -1<x<0 \\ 1-x, & 0\le x<1 \\ 0, & 1<x<2 \end{cases}$

Problem 3. Find the half range sine series and half range cosine series for each function.

(a) $f(x)=x^2, \quad 0<x<\pi$

(b) $f(x)=4, \quad 0<x<1$

(c) $f(x)=\begin{cases} 2x, & 0\le x<\frac{1}{2} \\ 1, & \frac{1}{2}\le x<1 \end{cases}.$

(d) $f(x)=e^{-x}, \quad 0<x<2$

Thin rod of length p. Adapted from [2].

Problem 4. **(Challenge Problem)** A classic problem in the study of partial differential equation is that of the heat flow in a thin rod. Suppose that we have a very thin rod of length p that is perfectly insulated on its lateral surface (not necessarily at its ends). We can impose a coordinate system by placing the rod on the x-axis with one end at $x = 0$ and the other at $x = p$. We assume that the temperature is constant at each point on a given cross section so that the rod is modeled by a line segment. Let $u(x, t)$ be the temperature of the rod at the point x units from the left end at the time t. If the rod is uniform, it can be shown that u satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < p \quad t > 0$$

where the number a^2 depends on the thermal properties of the rod. To completely describe the state of the rod, we require some additional information. We need to know the initial temperature distribution (an initial condition when $t = 0$)

$$u(x, 0) = f(x), \quad \text{for} \quad 0 < x < p,$$

as well as the temperature at each end (boundary conditions at $x = 0$ and $x = p$). Here, we'll assume that the temperature at each end is fixed at zero (e.g. 0° C).

$$u(0, t) = 0, \quad u(p, t) = 0 \quad \text{for all} \quad t > 0.$$

In this exercise, we'll take $p = \pi$, $a^2 = 1$ and assume the initial temperature distribution shown in Figure1.1, and consider the *initial, boundary value problem*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi \quad t > 0 \\ u(0, t) = 0, \quad u(\pi, t) &= 0 \quad \text{for all} \quad t > 0 \\ u(x, 0) &= x(\pi - x), \quad \text{for} \quad 0 < x < \pi \end{aligned} \tag{1.8}$$

It can be shown that the series $u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$ is a formal solution[6] to the partial differential equation. Use the following steps to verify the validity of this series and to construct the complete formal solution to the problem 1.8.

[6]There are convergence issues to be considered when attempting to extend the principle of superposition to infinitely many terms and when applying term by term differentiation to an infinite series. Due to the negative exponential and nice boundedness property of the sine function, this series is convergent at each point (x, t).

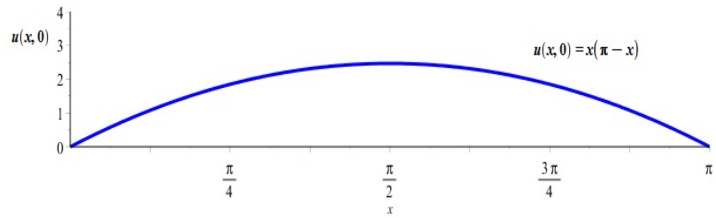


Figure 1.1: Initial temperature distribution.

(a) For each fixed $n = 1, 2, \ldots$, let $u_n(x,t) = b_n e^{-n^2 t}\sin(nx)$ where b_n is an arbitrary constant. Show that for any constant b_n

$$\frac{\partial u_n}{\partial t} = \frac{\partial^2 u_n}{\partial x^2}.$$

(b) Show that for each n, the function $u_n(x,t)$ defined above satisfies both boundary conditions, $u_n(0,t) = u_n(\pi,t) = 0$.

(c) Now we'll assume that the principle of superposition can be applied to the set of functions u_n for each $n = 1, 2, \ldots$ that you have established solve both the differential equation and the boundary conditions. Set

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx).$$

The last step is to apply the initial condition. In the series, set $t = 0$, and set the resulting series to $f(x) = x(\pi - x)$. Find the coefficients b_n, and write out the series $u(x,t)$. (Hint: You should see the makings of a half-range sine series for f.)

Chapter 2

Solutions

Definitions for all glossary items can be found in the lecture notes or by entering them into your favorite search engine. (You may find the occassional discrepancy in the use of terms. When in doubt, refer to how they are used in the current class.)

2.1 Section 1: Concepts and Terminology

Go to problems 1.1. **⟵ That number, 1.1, is a hyperlink back to the exercises.**

Get Acquainted Problem 1.

(a) $y' = 1 - \dfrac{1}{x^2} - \dfrac{1}{x}$ and $y'' = \dfrac{2}{x^3} + \dfrac{1}{x^2}$

(b)

$$\begin{aligned} x^2y'' + xy' - y &= x^2\left(\frac{2}{x^3} + \frac{1}{x^2}\right) + x\left(1 - \frac{1}{x^2} - \frac{1}{x}\right) - \left(x + \frac{1}{x} - \ln(x)\right) \\ &= \frac{2}{x} + 1 + x - \frac{1}{x} - 1 - x - \frac{1}{x} + \ln(x) \\ &= \ln(x) \end{aligned}$$

(c) They are the same.

(d) The function was at least two times differentiable (the derivatives could be obtained), and when substituted into the equation as y, the equation reduced to an identity. In particular, $\ln(x) = \ln(x)$.

Problem 1.

(a) (i) x, (ii) y, (iii) 3, (iv) Ordinary

(b) (i) x, y, and t, (ii) u, (iii) 2, (iv) Partial

(c) (i) t, (ii) x, (iii) 2, (iv) Ordinary

(d) (i) t, (ii) x and y, (iii) 2, (iv) Ordinary

Problem 2.

(a) Linear

(b) Nonlinear, due to $3x\dfrac{dx}{dt}$

(c) Nonlinear, due to xy^4

(d) Linear

(e) Linear

Problem 3.

(a) $y = \ln|x| + C, \quad 0 < x < \infty$ (Any interval that does not contain zero.)

(b) $y = c_1 x + c_2, \quad \mathbb{R}$

(c) $x = \sin^{-1} t + C, \quad -1 < t < 1$

(d) $y = \dfrac{x^4}{24} + \dfrac{1}{8}e^{2x} + c_1 x^2 + c_2 x + c_3, \quad \mathbb{R}$

Problem 4. All four figures exhibit characteristics that preclude the graph from being the graph of a solution to a differential equation (as we have defined it in the classical sense). (A), (C), and (D) have domains that are not an interval. The plot in (B) shows a point at which the function would fail to be differentiable (corner in the vicinity of $x = -1$.)

Remark: In a *weaker* sense, (B) could be considered a solution to a differential equation. For some models, it may be useful to overlook a differentiability anomoly at an isolated point like that.

Problem 5.

(a) use substitution

(b) use substitution or implicit differentiation

(c) use substitution

(d) use implicit differentiation

Problem 6.

(a) $m = 3$ and $m = 4$

(b) $m = 2 \pm \sqrt{5}$

Problem 7.

(a) $m = -2$ and $m = 4$

(b) $m = 3 \pm \sqrt{2}$

Problem 8.

(a) $Lx = 0$

(b) $Lx^2 = 3x^2$

(c) $Lx^{-1} = 0$

Problem 9. **(Challenge Problem)**
Part 1

(a)

$$\begin{aligned}
L(y+z) &= a_2(x)\frac{d^2(y+z)}{dx^2} + a_1(x)\frac{d(y+z)}{dx} + a_0(x)(y+z) \\
&= a_2(x)\left(\frac{d^2y}{dx^2} + \frac{d^2z}{dx^2}\right) + a_1(x)\left(\frac{dy}{dx} + \frac{dz}{dx}\right) + a_0(x)y + a_0(x)z \\
&= a_2(x)\frac{d^2y}{dx^2} + a_2(x)\frac{d^2z}{dx^2} + a_1(x)\frac{dy}{dx} + a_1(x)\frac{dz}{dx} + a_0(x)y + a_0(x)z \\
&= \left(a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y\right) + \left(a_2(x)\frac{d^2z}{dx^2} + a_1(x)\frac{dz}{dx} + a_0(x)z\right) \\
&= Ly + Lz
\end{aligned}$$

(b)

$$
\begin{aligned}
L(cy) &= a_2(x)\frac{d^2cy}{dx^2} + a_1(x)\frac{dcy}{dx} + a_0(x)cy \\
&= ca_2(x)\frac{d^2y}{dx^2} + ca_1(x)\frac{dy}{dx} + ca_0(x)y \\
&= c\left(a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y\right) \\
&= cLy
\end{aligned}
$$

Part 2

(a)

$$
\begin{aligned}
N(y+z) &= (y+z)\frac{d(y+z)}{dx} \\
&= y\frac{dy}{dx} + z\frac{dz}{dx} + y\frac{dz}{dx} + z\frac{dy}{dx} \\
&= Ny + Nz + y\frac{dz}{dx} + z\frac{dy}{dx}
\end{aligned}
$$

The remaining term $y\frac{dz}{dx} + z\frac{dy}{dx}$ may be zero for special choices of y and z. But it is not zero for all possible choices (try for example $y = 1$, $z = x$ defined for all real x).

(b)

$$
N(cy) = (cy)\frac{d(cy)}{dx} = c^2y\frac{dy}{dx}.
$$

Since $c \neq c^2$ when c is different from 1 and 0, this will not equal cNy for any nonconstant function y

2.2 Section 2: Initial Value Problems

Go to problems 1.2

Get Acquainted Problem 2.

(a) $s(t) = \frac{1}{2\pi^2}t^2 + \cos t + C$

(b) $s(t) = \frac{1}{2\pi^2}t^2 + \cos t - 1$

(c) $s(1) = \frac{1}{2\pi^2} + \cos 1 - 1 \approx -0.41$ The ant is about 0.41 inches to the left of zero.

Problem 1.

(a) $y = xe^x - e^x + 3$

(b) $y = -\cos x - x + 2$

(c) $x = \sin^{-1} t - \dfrac{\pi}{6}$

Problem 2.

(a) $y = \dfrac{14}{5}x^2 + \dfrac{6}{5x^3}$

(b) $y = \dfrac{3}{20}x^2 + \dfrac{16}{5x^3}$

Problem 3.

(a) $y(t) = 1$

(b) If $y(t) < 1$, then $1 - y(t) > 0$. Since the derivative is positive, y will increase toward 1. If $y(t) > 1$, then $1 - y(t) < 0$. Since the derivative is negative, y will decrease towards 1. (In other words, when y is below 1 it goes up, and when it's above 1 it goes down. If it's exactly at 1, the derivative is zero and it stays there.)

Problem 4.

(a) Substitute

(b) Substitute

(c) $y = 2e^{-x} - xe^{-x} + 4$

(d) It's obvious because it's just the special case $c_3 = 0$ already considered. The first two conditions would require $c_1 = 1$, $c_2 = 3$. Then the third condition would result in the false statement " $-5 = 3$".

Problem 5.

(a) $y = C \sin x$ solves the BVP for any choice of C.

(b) There is no choice for c_1, c_2 such that "$0 = 1$".

(c) The only solution is $y = 0$ for all x in $(0, \pi/4)$.

Problem 6.

(a) $y_2 = 1.00100$

(b) $y_4 = 1.00175$

Problem 7.

(a) $y_2 = 0.34000$

(b) $y_4 = 0.32140$

Problem 8. $k = 8$

Problem 9. (Challenge Problem)

(a) Substitute into the ODE and show that the initial condition is satisfied.

(b) $y = 0$ for all x is a solution

(c) Substitute into the ODE and verify that the initial condition holds. Use the definition $y_c'(c) = \lim_{h\to 0} \frac{y_c(c+h)-y_c(c)}{h}$ as needed.

(d) No, there's no value of c that would make that piecewise defined function be the constant function $y = 0$. (You can kind of think of $y = 0$ being what you'd get if you pushed $c \to \infty$, but there's no real number c for which y_c would just be that constant function.)

2.3 Section 3: First Order Equations: Separation of Variables

Go to problems 1.3

Get Acquainted Problem 3.

(a) $\frac{1}{\sqrt{y}}\frac{dy}{dx}\,dx = x^2\,dx$

(b) $\int y^{-1/2}\,dy = \int x^2\,dx \quad \Longrightarrow \quad 2y^{1/2} = \frac{x^3}{3} + C$

(c) $C = 2$ The solution is given implicitly by

$$2\sqrt{y} = \frac{x^3}{3} + 2 \quad \text{and explicitly by} \quad y = \left(\frac{x^3}{6} + 1\right)^2$$

(d) We assumed that $y \neq 0$—that we were not dividing by zero. Since the initial y value is 1, and y is continuous on some interval (it's differentiable), there is an interval over which y should be bounded away from zero.

Problem 1.

(a) $y = Ke^{x^2/2}$,

(b) $y = \frac{1}{x+c}$, or $y = 0$

(c) This equation is not separable.

(d) $x = Ke^{t^2/2+2t} - 1$

(e) Implicit solutions are given by $\ln|u-1| - \ln|u+1| = t^2 + c$ or $u = \pm 1$, explicit solutions are $u = \frac{Ke^{t^2}+1}{1-Ke^{t^2}}$ or $u = -1$

(f) $y = 1 - Ke^{-x^2/2}$

(g) This equation is not separable.

Problem 2.

(a) $y = e^{-1/2}e^{x^2/2}$,

(b) $y = \tan\left(\tan^{-1} t + \frac{\pi}{4}\right)$

(c) $2x^2 \ln x - x^2 = 4t^3 + e^2$

(d) $u = \dfrac{1 - e^{t^2}}{1 + e^{t^2}}$

(e) $y = \tan^{-1}\left(\frac{t}{2} + \frac{1}{4}\sin(2t) + 1\right)$

Problem 3. $P(t) = \dfrac{MP_0}{(M - P_0)e^{-Mt} + P_0}$, in the limit $P(t) \to M$ as $t \to \infty$.

Notice that from the ODE, $\dfrac{dP}{dt} = P(M - P)$, with M positive, that $P'(t)$ is positive if $0 < P < M$, and $P'(t) < 0$ if $P > M$. So $P(t)$ increases when P is less than M and decreases when P is bigger than M.

Problem 4. $r(t) = \dfrac{3\alpha}{\beta} + \left(r_0 - \dfrac{3\alpha}{\beta}\right) e^{-\frac{\beta}{3\rho}t}$. The exponential factor goes to zero as $t \to \infty$ giving the limit $r \to \dfrac{3\alpha}{\beta}$.

Problem 5. Take the derivative using the Fundamental Theorem of Calculus (FTC), and verify that the initial condition is satisfied.

Problem 6. **(Challenge Problem)** Use the FTC with the chain rule, and verify that the initial condition is satisfied.

2.4 Section 4: First Order Equations: Linear & Special

Go to problems 1.4

Get Acquainted Problem 4.

(a) $e^{2x}\dfrac{dy}{dx} + 2e^{2x}y = 18e^{2x}$

(b) $\dfrac{d}{dx}\left[e^{2x}y(x)\right] = e^{2x}\dfrac{dy}{dx} + 2e^{2x}y$

(c) They're clearly the same thing. $\dfrac{d}{dx}\left[e^{2x}y\right] = 18e^{2x}$

(d) $y = 9 + Ce^{-2x}$

(e) Do the substitution.

Problem 1. For each of these, you should find that $\frac{d}{dx}(\mu(x)y) = \mu\frac{dy}{dx} + \mu P(x)y$.

(a) $\mu = \sec x$

(b) $\mu = x^{-1}$

(c) $\mu = (x+1)e^{-x}$

Problem 2.

(a) $y = \dfrac{2x^3 + C}{x}$

(b) $y = -\cot x + C \csc x$

(c) $x = te^{-t}\ln t - te^{-t} + Ce^{-t}$

(d) $u = \dfrac{\theta - \cos\theta + C}{\sec\theta + \tan\theta}$

(e) $y = t\sin t + Ct$

Problem 3.

(a) $y = \dfrac{2x^3 - 2}{x}$

(b) $x = A + (x_0 - A)e^{-kt}$

(c) $u = \dfrac{\theta - \cos\theta + 1}{\sec\theta + \tan\theta}$

(d) $r(t) = \dfrac{3\alpha}{\beta} + \left(r_0 - \dfrac{3\alpha}{\beta}\right)e^{-\frac{\beta}{3\rho}t}$

Problem 4. $q(t) = E_0C_0\left(1 - e^{-\frac{t}{R_0C_0}}\right)$ The steady state is $q_p(t) = E_0C_0$ and the transient state is $q_c(t) = -E_0C_0e^{-\frac{t}{R_0C_0}}$

Problem 5. Answers will vary.

Problem 6. $y = \frac{2}{\sqrt{\pi}} e^{x^2} \int_0^x e^{-t^2}\, dt = e^{x^2} \text{erf}(x)$

Problem 7.

$$y(t) = \begin{cases} 1 - e^{-t}, & 0 \le t < 3 \\ (e^3 - 1)e^{-t}, & t \ge 3 \end{cases}$$

The solution is continuous at 3. Note that

$$\lim_{t \to 3^-} y(t) = \lim_{t \to 3^+} y(t) = 1 - e^{-3}$$

2.4.1 Some Special First Order Equations

Go to problems 1.4.1

Problem 8.

(a) $x^2 y^2 + 6x - 7y = C$

(b) $e^x \cos y + e^y \sin x = C$

Problem 9.

(a) $\mu = x, \quad x^2 y^3 + x^3 y = C$

(b) $\mu = y^3, \quad x^2 y^4 + y^6 - 10y^4 = C$

(c) $\mu = e^{-x}, \quad e^{-x} y^2 + \cos y = C$

Problem 10.

(a) $(P(x)y - f(x))\, dx + \, dy = 0$

(b) $\frac{\partial M}{\partial y} = P(x) \neq 0 = \frac{\partial N}{\partial x}$

(c) $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{P(x) - 0}{1} = P(x)$ so that $\mu = \exp\left(\int P(x)\, dx\right)$

(d) This is the traditional integrating factor.

Problem 11. $M(x, y) = ye^{xy} + 2xy^3 + f(x)$

Problem 12. $x^2y^2 + \sin^2(x) + y^2 = 9$

Problem 13.

(a) $y = \dfrac{1}{cx - x^2}$

(b) $y = \sqrt[3]{1 + cx^{-3}}$

(c) $y = \left(1 + ce^{-3x/2}\right)^{2/3}$

Problem 14.

(a) $y = (11x^3 - 3x^4)^{-1/3}$

(b) $y = (2e^x - 1)^2$
Care has to be taken when obtaining this last solution. One obtains the equation $\sqrt{y} = Ce^x - 1$. Applying the initial condition gives $C = 2$. If one squares first, to obtain $y = (Ce^x - 1)^2$, it appears as though C can equal 2 or zero. Both would seem to result in $y = 1$ when $x = 0$. However, this would lead to a false statement $\sqrt{y} = 0 \cdot e^x - 1$. The constant function $y(x) = 1$ doesn't solve the original ODE as one can easily check. This is a case in which squaring produces a superfluous solution that does not actually solve the original equation.

2.5 Section 5: First Order Equations: Models and Applications

Go to problems 1.5

Get Acquainted Problem 5.

(a) $\dfrac{dQ}{dt} = 300 - \dfrac{1}{30}Q$

(b) $Q(t) = 9000 - 8950e^{-t/30}$

(c) 9000 pounds

Problem 1. $q(t) = E_0C_0(1 - e^{-t/R_0C_0}), \quad q \to E_0C_0 \quad \text{as} \quad t \to \infty.$

Problem 2.

$$i(t) = \frac{1}{95}e^{-t} + \frac{94}{95}e^{-20t}$$

Problem 3.

$$\frac{dA}{dt} + \frac{1}{100}A = 12, \quad A(0) = 10. \qquad A(t) = 1200 - 1190e^{-t/100}$$

Problem 4.

$$\frac{dA}{dt} + \frac{5}{400-t}A = 12, \quad A(0) = 10. \quad A(t) = 3(400-t) - \frac{1190}{400^5}(400-t)^5, \quad 0 \le t \le 400$$

Problem 5. The concentration is $\frac{468.75}{1000}$ lb/gal

Problem 6. If $x(t)$ is the mass of pollutant at time t, then

$$x(t) = 10(100-t) - \frac{(100-t)^4}{10^5}.$$

There's 437.5 grams of pollutant at $t = 50$ when the tank is half empty.

Problem 7. The ODE in question is both linear and separable. Letting $x(t)$ be the salt in pounds,

$$\frac{dx}{dt} = -\frac{1}{250}x, \quad x(0) = 150.$$

The solution

$$x(t) = 150e^{-t/250}.$$

The concentration, $\frac{x(t)}{500}$ lbs per gallon, is 0.15 when $t = 250\ln(2)$. This is roughly 173 minutes.

Problem 8.

$$i(t) = \frac{E_0}{R_0} + \left(i_0 - \frac{E_0}{R_0}\right)e^{-R_0t/L_0}, \quad \lim_{t\to\infty} i(t) = \frac{E_0}{R_0} + \left(i_0 - \frac{E_0}{R_0}\right)\cdot 0 = \frac{E_0}{R_0}$$

Problem 9.

(a) α is in units of *per minute*

(b) $\frac{dV}{dt} = 10 - \alpha V \quad V(0) = 0$

(c) $V(t) = 20(1 - e^{-t/2})$

(d) $\frac{dV}{dt} = 10e^{-t/2} > 0$, for all $t > 0$. Hence V is strictly increasing on $(0, \infty)$. Moreover, $V \to 20$ as $t \to \infty$. Thus V will never exceed 20 gallons (but can be arbitrarily close to 20 gallons). The minimum tank capacity is therefore 20 gallons.

Problem 10.

(a)

$$x(t) = \frac{kA_1}{k^2 + \omega^2}(k\cos(\omega t) + \omega \sin(\omega t)) + A_0 + Ce^{-kt}$$

(b) The initial conditions would determine the value of C. But for any value of C, $Ce^{-kt} \to 0$ as $t \to \infty$. So this last term in the sum becomes negligible (how fast or slow depends on k, but exponentials tend to die pretty quickly). So no, the initial conditions don't contribute much to the long term solution which will oscillate about A_0.

Problem 11.

$$I(t) = \frac{SI_0}{(S - I_0)e^{-rSt} + I_0}, \qquad \lim_{t\to\infty} I(t) = S$$

Problem 12.

$$P(t) = \frac{P_0}{1 - P_0 t}, \qquad \lim_{t \to \frac{1}{P_0}^-} P(t) = \infty$$

Notice that the solution P exibits *blow up* (i.e. tends to infinity) in finite time.

Problem 13.

(a) The equilibria on the indicated interval are -4, -2, 0, 2, and 4. More generally, $\sin\left(\frac{\pi x}{2}\right) = 0$ whenever x is an even integer.

(b) -2 and 2 are stable, and -4, 0, 4 are unstable. More generally, if x_0 is an even integer that is not a multiple of 4, then x_0 is a stable equilibrium, and if x_0 is a multiple of 4, it is an unstable equilibrium.

(c)

$$\text{(i) } x \to -2, \quad \text{(ii) } x \to -2, \quad \text{(iii) } x(t) = 0, \quad \text{(iv) } x \to 2, \quad \text{(v) } x \to 2.$$

2.6 Section 6: Linear Equations: Theory and Terminology

Go to problems 1.6

Get Acquainted Problem 6.

(a) Substitute into the ODE and use the Wronskian to verify linear independence.

(b) $y = c_1 e^{2x} + c_2 e^{5x}$

(c) $y = e^{5x} - 2e^{2x}$

(d) $y = \left(\dfrac{5k_0 - k_1}{3} \right) e^{2x} + \left(\dfrac{k_1 - 2k_0}{3} \right) e^{5x}$

Problem 1.

(a) $W = -\dfrac{6}{x}$

(b) $W = 2$

(c) $W = -2\alpha$

(d) $W = be^{2ax}$

(e) $W = (n - m)x^{m+n-1}$

(f) $W = e^{2ax}$

Problem 2.

(a) They are linearly dependent.

(b) They are linearly independent.

(c) They are linearly dependent. Recall $\cos(2x) = 1 - 2\sin^2 x$.

(d) They are linearly independent if $\alpha \neq \beta$.

(e) They are linearly independent if $m \neq n$

Problem 3.

(a) Substitute

(b) $y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$

(c) $y = k_0 \cos(\ln x) + k_1 \sin(\ln x)$

Problem 4. This is the principle of superposition. Substitute to show that z_1 and z_2 solve the ODE. With a little effort, it can be shown that the Wronskian $W(z_1, z_2)(x) = -\frac{1}{2}W(y_1, y_2)(x)$. So the linear independence of $\{y_1, y_2\}$ ensures the linear independence of $\{z_1, z_2\}$.

Problem 5.

(a) Substitute and use the Wronskian.

(b) Substitute

(c) Substitute

(d) $y = c_1 e^{3x} + c_2 e^{2x} + x + 1 + 2e^{4x}$

(e) $y = 3e^{3x} - 6e^{2x} + x + 1 + 2e^{4x}$

Problem 6.

(a) Substitute and use the Wronskian.

(b) Substitute

(c) Substitute

(d) $y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + x + \ln x$

(e) $y = \cos(\ln x) - 2\sin(\ln x) + x + \ln x$

Problem 7. **(Challenge Problem)**

(a) $W(y_1, y_2)(x) = y_1 y_2' - y_1' y_2$, so that $\frac{dW}{dx} = y_1 y_2'' + y_1' y_2' - y_1' y_2' - y_1'' y_2 = y_1 y_2'' - y_1'' y_2$. Since y_1 and y_2 solve the original ODE, we have the system of equations

$$\begin{array}{ccccccc} y_2'' & + & P(x)y_2' & + & Q(x)y_2 & = & 0 \\ y_1'' & + & P(x)y_1' & + & Q(x)y_1 & = & 0 \end{array}$$

Multiply the first equation by y_1, the second by y_2 and subtract.

$$\begin{array}{rcccccl} y_1y_2'' & + & P(x)y_1y_2' & + & Q(x)y_1y_2 & = & 0 \\ y_2y_1'' & + & P(x)y_2y_1' & + & Q(x)y_2y_1 & = & 0 \\ \hline \underbrace{y_1y_2''-y_1''y_2}_{\frac{dW}{dx}} & + & P(x)\underbrace{(y_1y_2'-y_1'y_2)}_{W} & + & Q(x)\underbrace{(y_1y_2-y_1y_2)}_{0} & = & 0 \end{array}$$

That is,

$$\frac{dW}{dx} + P(x)W = 0. \tag{2.1}$$

(b) Consider the integrating factor $\mu = \exp\left(\int_{x_0}^{x} P(t)\,dt\right)$. Multiply (2.1) through by μ to obtain

$$\frac{d}{dx}\left(\exp\left(\int_{x_0}^{x} P(t)\,dt\right)W\right) = 0 \quad\Longrightarrow\quad \exp\left(\int_{x_0}^{x} P(t)\,dt\right)W = \text{constant}.$$

Calling this constant C, we have $W = C\exp\left(-\int_{x_0}^{x} P(t)\,dt\right)$. Apply $W(x_0) = C_0$.

(c) Compute the Wronskian $W(x^2, x^3+1)(x) = x(x^3-2)$. Note that $W(x^2, x^3+1)(0) = 0$ whereas $W(x^2, x^3+1)(1) = -1$. The Wronskian is neither zero nor never-zero on $(-2, 2)$. Hence these functions do not form a fundamental solution set for an equation (1.2) with P and Q continuous on the interval $(-2, 2)$.

2.7 Section 7: Reduction of Order

Go to problems 1.7

Get Acquainted Problem 7.

(a) Substitute

(b) $y_2' = 3x^2u + x^3u'$, $\quad y_2'' = 6xu + 6x^2u' + x^3u''$

(c) The equation for u simplifies to $u'' = 0$.

(d) $u(x) = c_1x + c_2$

(e) Substitute and use the Wronskian

(f) $y = k_1x^3 + k_2x^4$ There are two parameters which is expected for a second order ODE.

Problem 1.

- The equation must be linear and homogeneous.

- One solution of the equation must be known.

Problem 2.

(a) Substitute

(b) $y_2' = u'e^{-4x} - 4ue^{-4x}, \quad y_2'' = u''e^{-4x} - 8u'e^{-4x} + 16ue^{-4x}$

(c) The equation for u simplifies to $u'' - 8u' = 0$.

(d) $w(x) = ke^{8x}$ where k is any real number

(e) $u = \frac{k}{8}e^{8x} + C$

(f) $y = k_1 e^{-4x} + k_2 e^{4x}$ There are two parameters which is expected for a second order ODE.

Problem 3.

(a) $y_2 = \frac{1}{2}e^{3x}$

(b) This doesn't have the constant factor $\frac{1}{2}$. The principle of superposition states that if y is a solution to the linear, homogeneous equation, then so is cy for any choice of the constant c. Choosing $c = 2$ allows us to ignore the constant factor and obtain a simpler looking function for the solution set.

Problem 4.

(a) $y_2 = e^{4x}, \quad y = c_1 e^x + c_2 e^{4x}$

(b) $y_2 = x + 1, \quad y = c_1 e^x + c_2(x + 1)$

(c) $y_2 = \sin(\ln x), \quad y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$

(d) $y_2 = x^2 \ln x, \quad y = c_1 x^2 + c_2 x^2 \ln x$

Problem 5.

(a) The Wronskian $W(f_1, f_2)(x) = e^{2hx}$ which is nonzero for all x in $(-\infty, \infty)$.

(b) Substitute

(c) $y_2 = xe^{hx}$

(d) $y = e^{-4x} + 3xe^{-4x}$

Problem 6.

(a) The Wronskian $W(f_1, f_2)(x) = x^{2k-1}$ which is nonzero for all x in $(0, \infty)$.

(b) Substitute

(c) $y_2 = x^k \ln x$

(d) $y = 2x^3 - 8x^3 \ln x$

Problem 7. $y_2 = \sqrt{1-x^2}$, $\quad y = c_1 x + c_1\sqrt{1-x^2}$ and $y = 2\sqrt{1-x^2} - x$

2.8 Section 8: Homogeneous Equations with Constant Coefficients

Go to problems 1.8

Get Acquainted Problem 8.

(a) $y' = me^{mx}$ and $y'' = m^2 e^{mx}$

(b) $e^{mx}(m^2 - 7m + 10) = 0$

(c) $m_1 = 2$ and $m_2 = 5$

(d) Substitute

Problem 1.

(a) $m^2 - 2m - 3$

(b) $m^6 + 4m^5 - m^3 + 6m^2 - m + 2$

(c) This equation is not constant coefficient. It doesn't have a characteristic polynomial.

(d) $m^3 - 12m^2 + 48m - 64$

(e) This equation is neither linear, nor constant coefficient. The coefficient of y' depends on y. (The equation is easily solved! What type of equation is it?)

Problem 2.

(a) $m^2 - 2hm + h^2 = 0$, $m = h$

(b) The general solution is a linear combination of the solutions in a fundamental solution set. For a second order ODE, this requires two linearly independent functions.

(c) This leads to the inconsistent system of equations $c_1 + c_2 = 1$ and $c_1 + c_2 = 2$.

Problem 3.

(a) $y = c_1 e^{-t} + c_2 e^{3t}$

(b) $y = c_1 e^{-(2+\sqrt{3})x} + c_2 e^{-(2-\sqrt{3})x}$

(c) $y = c_1 e^{-2x} \cos(\sqrt{3}x) + c_2 e^{-2x} \sin(\sqrt{3}x)$

(d) $x = c_1 e^{t/3} + c_2 t e^{t/3}$

(e) $q = c_1 e^{-10t} \cos(20t) + c_2 e^{-10t} \sin(20t)$

(f) $x = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

(g) $x = c_1 e^{\lambda t} + c_2 e^{-\lambda t}$

Problem 4.

(a) $y = \frac{1}{4} e^{-t} - \frac{1}{4} e^{3t}$

(b) $y = 3e^{-2x} \cos(2x) + 4e^{-2x} \sin(2x)$

(c) $y = e^x + xe^x$

(d) $y = \left(\frac{2-\sqrt{2}}{2}\right) e^{(1+\sqrt{2})t} + \left(\frac{2+\sqrt{2}}{2}\right) e^{(1-\sqrt{2})t}$

(e) $x = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$

(f) $x = \frac{1}{2\lambda} e^{\lambda t} - \frac{1}{2\lambda} e^{-\lambda t}$

Problem 5.

(a) $x = c_1 e^{4t} + c_2 t e^{4t} + c_3 t^2 e^{4t}$

(b) $y = c_1e^x + c_2\cos(2x) + c_3\sin(2x)$

(c) $y = c_1e^{3x} + c_2e^{-3x} + c_3\cos(3x) + c_4\sin(3x)$

(d) $y = c_1e^{-2x} + c_2e^{3x} + c_3 + c_4x$

Problem 6.

(a) $y_c = c_1e^x + c_2e^{3x}$

(b) Substitute

(c) $y = c_1e^x + c_2e^{3x} + 2e^{-2x}$

(d) $y = -2e^x + 3e^{3x} + 2e^{-2x}$

Problem 7.

(a) $y_c = c_1e^x + c_2\cos(2x) + c_3\sin(2x)$

(b) Substitute

(c) $y = c_1e^x + c_2\cos(2x) + c_3\sin(2x) - 2x - 2$

(d) $y = 2e^x - \cos(2x) + \dfrac{1}{2}\sin(2x) - 2x - 2$

Problem 8.

(a) Substitution into the ODE gives the quadratic equation $r^2 + (a-1)r + b = 0$

(b) $y = c_1x^2 + c_2x^{-3}$

(c) $y = \dfrac{6}{5}x^2 - \dfrac{1}{5x^3}$

Problem 9. (Challenge Problem)

(a) $\dfrac{du}{dz} = \dfrac{dy}{dz} = \dfrac{dx}{dz}\dfrac{dy}{dx} = e^z\dfrac{dy}{dx} = x\dfrac{dy}{dx}$

(b) $\dfrac{d^2u}{dz^2} = \dfrac{d}{dz}\left(x\dfrac{dy}{dx}\right) = \dfrac{dx}{dz}\dfrac{dy}{dx} + x\dfrac{d}{dz}\left(\dfrac{dy}{dx}\right) = x\dfrac{dy}{dx} + x\dfrac{dx}{dz}\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = x\dfrac{dy}{dx} + x^2\dfrac{d^2y}{dx^2}$

(c) $u''(z) + (a-1)u'(z) + bu(z) = 0$

(d) Note that $ze^{rz} = (\ln x)x^r$

  (i) $y = c_1x^2 + c_2x^4$

  (ii) $y = c_1x^2 + c_2x^2\ln x$

  (iii) $y = c_1\cos(3\ln x) + c_2\sin(3\ln x)$

2.9 Section 9: Method of Undetermined Coefficients

Go to problems 1.9

Get Acquainted Problem 9.

(a) Substitute

(b) $y_p = \dfrac{3}{49}e^{6x}$

(c) Substitute

Problem 1.

(a) The left side must be constant coefficient.

(b) The function must come from the classes allowed, namely polynomial, exponential (e^{ax}), sine and cosine ($\sin(kx)$ or $\cos(kx)$), sums or products of these types. These types of functions have derivatives that terminate, repeat, or cycle.

Problem 2.

(a) $y_p = \dfrac{2}{3}x^3e^{-x}$

(b) $y_p = -\frac{9}{25}\sin(2x) - \frac{12}{25}\cos(2x)$

(c) $y_p = \dfrac{x^2}{16} - \dfrac{1}{128}$

(d) $y_p = \dfrac{11}{265}e^{2x}\cos(3x) + \dfrac{12}{265}e^{2x}\sin(3x)$

(e) $y_p = -x^2 - 2x$

Problem 3.

(a) The two equations arise, $2A = 0$ and $-8A = 3$ which can't both be satisfied with one number A.

(b) $y_p = Ax + B$ is the correct form. A particular solution is $y_p = -\dfrac{3x}{8} - \dfrac{3}{32}$.

Problem 4.

(a) $y = c_1e^{-x} + c_2xe^{-x} + x - 2$

(b) $y = c_1e^x\cos(2x) + c_2e^x\sin(2x) - \dfrac{3}{13}e^{-2x}$

(c) $y = c_1e^x + c_2\cos(x) + c_3\sin(x) + \frac{1}{20}e^{3x}$

(d) $y = c_2e^{2x} + c_2e^{3x} - xe^{2x}$

(e) $y = c_1e^{2x} + c_2xe^{2x} + 2x^2e^{2x}$

(f) $y = c_1\cos(3x) + c_2\sin(3x) + \dfrac{x^2}{9} - \dfrac{x}{9} - \dfrac{2}{81}$

(g) $y = c_1e^x + c_2xe^x + 6\cos(2x) + 8\sin(2x)$

(h) $y = c_1 + c_2e^{-3x} + \dfrac{3}{2}e^x\sin x - \dfrac{5}{2}e^x\cos x$

(i) $y = c_1 + c_2e^{-3x} - 8xe^{-x} - 4e^{-x}$

Problem 5.

(a) $y = c_1e^{-x} + c_2xe^{-x} + 3 + \dfrac{1}{2}\cos x$

(b) $y = c_1e^x\cos(2x) + c_2e^x\sin(2x) + 10x^2 + 8x - \frac{4}{5} - \dfrac{5}{8}e^{3x}$

(c) $y = c_1e^x + c_2\cos(x) + c_3\sin(x) + 2x\cos(x) + 2x\sin(x) - 2x - 2$

(d) $y = c_2e^{2x} + c_2e^{3x} + \dfrac{1}{2}e^x + 2e^{4x} - e^{6x}$

Problem 6.

(a) $x = c_1\cos(\omega t) + c_2\sin(\omega t)$

(b) $x_p = A\cos(\gamma t) + B\sin(\gamma t)$

(c) $x_p = At\cos(\omega t) + Bt\sin(\omega t)$

(d) $x = c_1\cos(2t) + c_2\sin(2t) - \dfrac{F_0}{5}\cos(3t)$

(e) $x = c_1\cos(2t) + c_2\sin(2t) + \dfrac{F_0}{4}t\sin(2t)$

Problem 7.

(a) $y_p = Ae^x + B\cos(2x) + C\sin(2x) + (De^x\cos(2x) + Ee^x\sin(2x))x$

(b) $y_p = Ax^2 + Bx + C + (Dx^2 + Ex + F)x^2e^{-x}$

(c) $y_p = (Ax + B)x + Cxe^{-2x} + De^{2x}$

(d) $y_p = (A\sin x + B\cos x)x^2 + C\sin(2x) + D\cos(2x)$

2.10 Section 10: Variation of Parameters

Go to problems 1.10.

Get Acquainted Problem 10.

(a) $y_1 = e^{-x}$ and $y_2 = xe^{-x}$

(b) $W = e^{-2x}$

(c) $u_1(x) = -\frac{1}{2}x^2$ and $u_2(x) = x$

(d) $y_p = \frac{1}{2}x^2e^{-x}$

(e) $y = e^{-x} + 2xe^{-x} + \frac{1}{2}x^2e^{-x}$

Problem 1.

(a) $y_p = -\frac{1}{4}\cos(2x)\ln|\sec(2x) + \tan(2x)|$

(b) $y_p = -\frac{1}{4}\cos(2x)\ln|\sec(2x)| + \frac{x}{2}\sin(2x)$

(c) $y_p = -\frac{1}{2}e^x\ln(1 + x^2) + xe^x\tan^{-1}x$

(d) $y_p = \frac{4}{15}x^{5/2}e^{2x}$

(e) $y_p = \frac{x^3}{8}$

(f) $y_p = -\frac{2\ln x}{x} - \frac{2(\ln x)^2}{x}$

(g) $y_p = -4x^{5/2}$

(h) $y_p = x^3 \ln x$

(i) $y_p = \frac{1}{2}x^3(\ln x)^2$

(j) $y_p = -\frac{1}{2}\sqrt{x}\cos\sqrt{x}$

Problem 2.

(a) $y_c = c_1e^{-x} + c_2xe^{-x}$

(b) $y_{p_1} = 8x^{7/2}e^{-x}$

(c) $y_{p_2} = x - 2$

(d) $y = 2e^{-x} + 2xe^{-x} + 8x^{7/2}e^{-x} + x - 2$

(e) You avoid integration by parts on $\int x^2e^x\,dx$.

Problem 3. $y = \sin x - \cos x + \cos x \ln|\sec x| - x\sin x + 2$

2.11 Section 11: Linear Mechanical Equations

Go to problems 1.11

Get Acquainted Problem 11.

(a) $m = \frac{5}{16}$ slugs, and $k = 20$ lb/ft

(b) $\omega^2 = 64$ per sec^2

(c) $x(t) = \frac{1}{4}\cos(8t) - \frac{1}{4}\sin(8t)$

(d) The smallest positive t for which $x(t) = 0$ is when $8t = \frac{\pi}{4}$ so the time is $\frac{\pi}{32}$ seconds.

(e) The amplitude $A = \frac{\sqrt{2}}{4}$ feet.

Throughout these exercises, take the gravitational constant $g = 32$ ft/sec^2 or $g = 9.8$ m/sec^2 as appropriate.

Problem 1. $x(t) = -\cos(4\sqrt{6}t) + \dfrac{1}{2\sqrt{6}}\sin(4\sqrt{6}t)$ feet

Problem 2. $x(t) = -0.015\cos(14\sqrt{10}t)$ meters

Problem 3. $k = 30$ lb/ft and the 2 lb object has mass $m = \dfrac{1}{16}$ slugs.
$x(t) = \dfrac{1}{4\sqrt{30}}\sin(4\sqrt{30}t)$

Problem 4.

(a) $\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = \frac{a^2+b^2}{a^2+b^2} = 1$

(b) $f(t) = A\sin(\omega t + \phi)$

(c) $f(t) = A\cos(\omega t - \hat{\phi})$

Problem 5. $x(t) = \dfrac{2}{3}\cos(16t) - \dfrac{1}{4}\sin(16t) = \dfrac{\sqrt{73}}{12}\sin(16t + \phi)$ where $\phi = \cos^{-1}\left(-\frac{3}{\sqrt{73}}\right) \approx 1.93$. The amplitude $A = \dfrac{\sqrt{73}}{12}$, the period $T = \dfrac{2\pi}{16} = \dfrac{\pi}{8}$ and the phase shift $\phi \approx 110.6°$ when expressed in degrees.

Problem 6. $\sqrt{\frac{k}{m_1}} = 2\sqrt{\frac{k}{m_2}}$ hence the mass of the slower period system is four times the other mass.

Problem 7.

(a) $m = 2$ slugs

(b) $2x'' + 8x' + 26x = 0$, i.e. $x'' + 4x' + 13x = 0$. The system is underdamped with characteristic roots $r = -2 \pm 3i$.

(c) $x(t) = \frac{1}{2}e^{-2t}\cos(3t) + e^{-2t}\sin(3t)$

Problem 8.

(a) $x''+6x'+9x=0$ This is critically damped with the one root $r=-3$ to the characteristic equation.

(b) $x(t)=0.1e^{-3t}+0.8te^{-3t}$

(c) $x(t)=0.02e^{-3t}+0.5te^{-3t}+0.08\cos t+0.06\sin t$

Problem 9.

(a) $2x''+bx'+128x=0$

(b) The system is critically damped if $b^2=1024$ i.e. if $b=32$. The system is overdamped if $b>32$ and underdamped if $0<b<32$

Problem 10.

(a) The resonance $\gamma=\omega=5$

(b) $x(t)=\dfrac{1}{24}\cos t-\dfrac{1}{24}\cos(5t)$

(c) $x(t)=\dfrac{1}{10}t\sin(5t)$

Problem 11. $x(t)=\dfrac{F_0}{2\omega^2}\sin(\omega t)-\dfrac{F_0}{2\omega}t\cos(\omega t)$

2.12 Section 12: LRC Series Circuits

Go to problems 1.12

Get Acquainted Problem 12.

(a) $L_0\dfrac{d^2q}{dt^2}+R_0\dfrac{dq}{dt}+\dfrac{1}{C_0}q=E_0$

(b) $L_0\dfrac{d^2i}{dt^2}+R_0\dfrac{di}{dt}+\dfrac{1}{C_0}i=0$

Problem 1.

(a) $q(t) = 2e^{-20t}\cos(40t) + \frac{19}{20}e^{-20t}\sin(40t)$, $i(t) = -2e^{-20t}\cos(40t) - 99e^{-20t}\sin(40t)$

(b) $q(t) = 2e^{-10t}\cos(30t) + \frac{2}{3}e^{-10t}\sin(30t)$, $i(t) = -\frac{200}{3}e^{-10t}\sin(30t)$

(c) $q(t) = 3e^{-30t}\cos(40t) + 2e^{-30t}\sin(40t)$, $i(t) = -10e^{-30t}\cos(40t) - 180e^{-30t}\sin(40t)$

Problem 2. $q(t) = \frac{1}{101}\left(10e^{-t} - 10\cos(10t) + \sin(10t)\right)$

Problem 3. $q(t) = -2e^{-t} + 4te^{-t} + 2e^{-3t}$

Problem 4. $i_p(t) = -\frac{1}{3}\cos(10t) - \frac{2}{3}\sin(10t)$

2.13 Section 13: The Laplace Transform

Go to problems 1.13

Get Acquainted Problem 13.

(a) $\cos(2t)\sin(2t) = \frac{1}{2}\sin(4t)$

(b) $\mathscr{L}\{\sin(4t)\} = \frac{4}{s^2+16}$

(c) $\mathscr{L}\{t^2\} = \frac{2!}{s^3}$

(d) $\mathscr{L}\{y(t)\} = \frac{1}{2}\left(\frac{4}{s^2+16}\right) + 2\left(\frac{2!}{s^3}\right) = \frac{2}{s^2+16} + \frac{4}{s^3}$

Problem 1.

(a) $F(s) = \frac{1}{s-3} \quad s > 3$

(b) $F(s) = \frac{e^{-2}}{s-1} \quad s > 1$

(c) $F(s) = \frac{2}{s} - \frac{e^{-2s}}{s} \quad s > 0$

(d) $F(s) = \begin{cases} \dfrac{1}{s-1} - \dfrac{e^{-(s-1)}}{s-1}, & s \neq 1 \\ 1, & s = 1 \end{cases}$ (This is continuous at 1.)

Problem 2.

(a) $F(s) = \dfrac{3e^{-2s}}{s} - \dfrac{3e^{-4s}}{s}$ for $s > 0$ and $F(0) = 6$

(b) $F(s) = \dfrac{1}{s^2} - \dfrac{e^{-s}}{s^2}$ for $s > 0$

Problem 3.

(a) $F(s) = \dfrac{72}{s^5} - \dfrac{4}{s^3} + \dfrac{1}{s^2} + \dfrac{4}{s}$

(b) $F(s) = \dfrac{s}{s^2+9} + \dfrac{1}{2}\dfrac{3}{s^2+9}$

(c) $F(s) = \dfrac{a}{s^2} + \dfrac{b}{s} = \dfrac{a+bs}{s^2}$

(d) $F(s) = \dfrac{1}{2}\dfrac{1}{s-1} + \dfrac{1}{2}\dfrac{1}{s+1}$

(e) $F(s) = \dfrac{8}{s^3} - \dfrac{4}{s^2} + \dfrac{1}{s}$

(f) $F(s) = \dfrac{1}{s} - \dfrac{2}{s+3} + \dfrac{1}{s+6}$

(g) $F(s) = \dfrac{1}{s-2} - \dfrac{2}{s} + \dfrac{1}{s+2}$

(h) $F(s) = \dfrac{\alpha s}{s^2+\omega^2} + \dfrac{\beta\omega}{s^2+\omega^2}$

(i) $F(s) = \dfrac{\sqrt{3}}{2}\dfrac{1}{s^2+1} - \dfrac{1}{2}\dfrac{s}{s^2+1}$

(j) $F(s) = \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4\omega^2}\right)$

Problem 4.

(a) $\mathscr{L}\{t^2\} = \dfrac{2}{s^3}$ whereas $\mathscr{L}\{t\} \cdot \mathscr{L}\{t\} = \left(\dfrac{1}{s^2}\right)^2 = \dfrac{1}{s^4}$

(b) $\mathscr{L}\{\sin(2t)\} = \dfrac{2}{s^2+4}$ whereas $2\mathscr{L}\{\sin t\} \cdot \mathscr{L}\{\cos t\} = \dfrac{2s}{(s^2+1)^2}$

(c) $e^{at}e^{bt} = e^{(a+b)t}$

(d) $\mathscr{L}\{e^{at}e^{bt}\} = \dfrac{1}{s-(a+b)}$ whereas $\mathscr{L}\{e^{at}\} \cdot \mathscr{L}\{e^{bt}\} = \dfrac{1}{(s-a)(s-b)}$

Problem 5. **(Challenge Problem)**

(a) $\Gamma(r+1) = \int_0^\infty e^{-t}t^r\,dt$

(b) Let $\tau = st$ so that $dt = \frac{1}{s}d\tau$. When $t = 0$, $\tau = 0$, and since $s > 0$ we have $\tau \to \infty$ as $t \to \infty$. Making the substitution

$$\mathscr{L}\{t^r\} = \int_0^\infty e^{-st}t^r\,dt = \int_0^\infty e^{-\tau}\left(\frac{\tau}{s}\right)^r \frac{1}{s}\,d\tau = \frac{1}{s^{r+1}}\int_0^\infty e^{-\tau}\tau^r\,d\tau = \frac{\Gamma(r+1)}{s^{r+1}}$$

(c) Integrate by parts letting $u = t^r$ and $dv = e^{-t}\,dt$. Then $du = rt^{r-1}\,dt$ and $v = -e^{-t}$.

$$\Gamma(r+1) = \int_0^\infty e^{-t}t^r\,dt = -t^re^{-t}\Big|_0^\infty + r\int_0^\infty e^{-t}t^{r-1}\,dt = 0 + r\Gamma(r)$$

(d) (i) $F(s) = \dfrac{\sqrt{\pi}}{s^{1/2}}$

(ii) $F(s) = \dfrac{\sqrt{\pi}/2}{s^{3/2}}$

(iii) $F(s) = \dfrac{3\sqrt{\pi}/4}{s^{5/2}}$

2.14 Section 14: Inverse Laplace Transforms

Go to problems 1.14.

Get Acquainted Problem 14.

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3}\right\} = \dfrac{1}{2!}t^2$

(b) $\dfrac{1}{s^2-3s} = \dfrac{-\frac{1}{3}}{s} + \dfrac{\frac{1}{3}}{s-3}$

(c) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2-3s}\right\} = -\dfrac{1}{3} + \dfrac{1}{3}e^{3t}$

(d) $\mathscr{L}^{-1}\left\{\frac{1}{s^3}+\frac{1}{s^2-3s}\right\}=\frac{1}{2!}t^2-\frac{1}{3}+\frac{1}{3}e^{3t}$

Problem 1.

(a) $f(t)=t-e^{2t}$

(b) $f(t)=\cos(3t)+\frac{1}{3}\sin(3t)$

(c) $f(t)=3+4t+\frac{5}{2}t^2$

(d) $f(t)=\frac{1}{2}e^{3t}$

Problem 2. $\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\}=\frac{t^{n-1}}{(n-1)!},\quad n\geq 2$

Problem 3. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\}=\frac{1}{k}\sin(kt)$

Problem 4.

(a) $f(t)=\frac{1}{3}e^{t}-\frac{1}{3}e^{-2t}$

(b) $f(t)=\frac{3}{4}e^{-3t}+\frac{1}{4}e^{t}$

(c) $f(t)=-1+\cos t+\sin t$

(d) $f(t)=\frac{1}{3}e^{t}+e^{-t}-\frac{4}{3}e^{-2t}$

(e) $f(t)=\frac{1}{3}\sin t+\frac{1}{3}\cos t-\frac{1}{6}\sin(2t)-\frac{1}{3}\cos(2t)$

(f) $f(t)=\frac{1}{2}t^2+\frac{2}{3}\sin(3t)$

Problem 5.

(a) $\mathscr{L}\{\sinh(kt)\}=\frac{k}{s^2-k^2}$

(b) $\mathscr{L}\{\cosh(kt)\} = \dfrac{s}{s^2 - k^2}$

Problem 6.

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-1)(s+2)}\right\} = \dfrac{1}{3}e^t - \dfrac{1}{3}e^{-2t}$ whereas $\mathscr{L}^{-1}\left\{\dfrac{1}{s-1}\right\} \cdot \mathscr{L}^{-1}\left\{\dfrac{1}{s+2}\right\} = e^t \cdot e^{-2t} = e^{-t}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^5}\right\} = \dfrac{1}{24}t^4$ whereas $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2}\right\} \cdot \mathscr{L}^{-1}\left\{\dfrac{1}{s^3}\right\} = t \cdot \dfrac{1}{2}t^2 = \dfrac{1}{2}t^3$

Problem 7.

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-1)(s+2)}\right\} = \displaystyle\int_0^t e^\tau e^{-2(t-\tau)}\, d\tau = \dfrac{1}{3}e^t - \dfrac{1}{3}e^{-2t}$

(b) Note that $\dfrac{s}{(s^2+1)^2} = \left(\dfrac{s}{s^2+1}\right)\left(\dfrac{1}{s^2+1}\right)$. That's the product of the transforms of $f(t) = \cos t$ and $g(t) = \sin t$.
$\mathscr{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \displaystyle\int_0^t \cos\tau \sin(t-\tau)\, d\tau = \dfrac{1}{2}t\sin t$

Problem 8.

(a) $\mathscr{L}\left\{\displaystyle\int_0^t \cos(2\tau)\cos(4(t-\tau))\, d\tau\right\} = \mathscr{L}\{\cos(2t)\} \cdot \mathscr{L}\{\cos(4t)\} = \dfrac{s^2}{(s^2+4)(s^2+16)}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{4}{s(s^2+16)}\right\} = (\sin(4t) * 1)(t) = \displaystyle\int_0^t \sin(4\tau)\, d\tau = \dfrac{1}{4} - \dfrac{1}{4}\cos(4t)$

(c) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3(s-2)}\right\} = \left(\tfrac{1}{2}t^2 * e^{2t})\right) = \displaystyle\int_0^t \dfrac{1}{2}\tau^2 e^{2(t-\tau)}\, d\tau = \dfrac{1}{8}e^{2t} - \dfrac{1}{8} - \dfrac{1}{4}t - \dfrac{1}{4}t^2$

(d) **(Challenge Problem)** $\mathscr{L}\left\{\dfrac{1}{(s+1)(s-1)(s-2)}\right\} = \dfrac{1}{3}e^{2t} - \dfrac{1}{2}e^t + \dfrac{1}{6}e^{-t}$

Note that this requires two convolution steps. If we set

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t},$$

$$g(t) = \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t, \quad \text{and}$$

$$h(t) = \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t},$$

then our inverse transform will be $(f*(g*h))$. Compute $(g*h)(t)$, and then take the convolution of f with this result.

$$(g*h)(t) = e^{2t} - e^{t}, \quad \text{and} \quad \left(f*\left(e^{2t}-e^{t}\right)\right)(t) = \frac{1}{3}e^{2t} - \frac{1}{2}e^{t} + \frac{1}{6}e^{-t}$$

Problem 9.

(a) $(f*g)(t) = \int_0^t \tau e^{2(t-\tau)}\,d\tau = -\frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{2t} = \int_0^t e^{2\tau}(t-\tau)\,d\tau = (g*f)(t).$

(b) $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$. Let $u = t-\tau$ so that $\tau = t-u$. Then $du = -d\tau$. When $\tau = 0$, $u = t$, and when $\tau = t$, $u = 0$. Applying this change of variables

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = -\int_t^0 f(t-u)g(u)\,du = \int_0^t g(u)f(t-u)\,du = (g*f)(t).$$

(c) $(t*1)(t) = \int_0^t \tau(1)\,d\tau = \left.\frac{\tau^2}{2}\right|_0^t = \frac{t^2}{2}.$ $\quad (t*1)(t) = \frac{t^2}{2} \neq t$

(d) The previous example seems to indicate that there is a derivative/antiderivative relationship between f and $f*1$. Suppose $f(t)$ is continuous on $[0,\infty)$ and let $F(t) = \int_0^t f(\tau)\,d\tau = (f*1)(t)$. By the Fundamental Thm of Calculus, $F'(t) = f(t)$.

Problem 10. (Challenge Problem)

$$f(t) = 2 + e^{2t}$$

2.15 Section 15: Shift Theorems

Go to problems 1.15

Get Acquainted Problem 15.

(Part 1)

(a) $s^2 - 4s + 8 = (s-2)^2 + 4$

(b) $\frac{s}{(s-2)^2+4} = \frac{s-2}{(s-2)^2+4} + \frac{2}{(s-2)^2+4}$

(c)

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s}{s^2-4s+8}\right\} &= \mathscr{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4}\right\}+\mathscr{L}^{-1}\left\{\frac{2}{(s-2)^2+4}\right\}\\ &= e^{2t}\cos(2t)+e^{2t}\sin(2t)\end{aligned}$$

(Part 2)

(a) $f(t) = t - t\mathscr{U}(t-2) + e^t\mathscr{U}(t-2)$

(b) $g_1(t+a) = t+a$ and $g_2(t+a) = e^{t+a} = e^t \cdot e^a$

(c)

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \mathscr{L}\{t - t\mathscr{U}(t-2) + e^t\mathscr{U}(t-2)\}\\ &= \mathscr{L}\{t\} - \mathscr{L}\{t\mathscr{U}(t-2)\} + \mathscr{L}\{e^t\mathscr{U}(t-2)\}\\ &= \frac{1}{s^2} - e^{-2s}\mathscr{L}\{t+2\} + e^{-2s}\mathscr{L}\{e^{t+2}\}\\ &= \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) + \frac{e^{-2s}e^2}{s-1}\end{aligned}$$

Problem 1.

(a) $F(s) = \dfrac{3}{(s+1)^2+9}$

(b) $F(s) = \dfrac{2}{(s-4)^2}$

(c) $F(s) = \dfrac{4!}{s^5} + \dfrac{2(2!)}{(s-1)^3} + \dfrac{1}{s-2}$

(d) $F(s) = \dfrac{6A}{(s-a)^4} + \dfrac{2B}{(s-a)^3} + \dfrac{C}{(s-a)^2} + \dfrac{D}{s-a}$

(e) $F(s) = \dfrac{A(s+\lambda)}{(s+\lambda)^2+\omega^2} + \dfrac{B\omega}{(s+\lambda)^2+\omega^2}$

Problem 2.

(a) $f(t) = \dfrac{1}{5!}e^{3t}t^5$

(b) $f(t) = -\dfrac{1}{9}e^t + \dfrac{1}{3}te^t + \dfrac{1}{9}e^{-2t}$

(c) $f(t) = e^{-3t}\cos(\sqrt{5}t) - \dfrac{3}{\sqrt{5}}e^{-3t}\sin(\sqrt{5}t)$

(d) $f(t) = e^{-2t} - 3te^{-2t} + \frac{1}{2}t^2e^{-2t}$

(e) $f(t) = \frac{1}{3}\cos(3t) + \frac{4}{9}\sin(3t) - \frac{1}{3}e^{-6t} - \frac{10}{3}te^{-6t}$

Problem 3.

(a) $F(s) = e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$

(b) $F(s) = \frac{e^{-s}}{s} + \frac{e^2e^{-s}}{s-2}$

(c) $F(s) = e^{-s/3}\left(\frac{\frac{1}{2}\pi}{s^2+\pi^2} + \frac{\frac{\sqrt{3}}{2}s}{s^2+\pi^2}\right)$

(d) $F(s) = \frac{e^{-\pi(s+1)}(s+1)}{(s+1)^2+2^2}$

(e) $F(s) = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$

Problem 4.

(a) $f(t) = \frac{1}{5}\sin\left(5\left(t-\frac{1}{2}\right)\right)\mathscr{U}\left(t-\frac{1}{2}\right)$

(b) $f(t) = \left(\frac{1}{4} - \frac{1}{4}e^{-4(t-2)}\right)\mathscr{U}(t-2)$

(c) $f(t) = (1-\cos(t-1))\,\mathscr{U}(t-1)$

(d) $f(t) = \frac{1}{3!}e^{5(t-2)}(t-2)^3\mathscr{U}(t-2)$

Problem 5.

(a) $\mathscr{L}\{\mathscr{U}(t-\beta)\} = \frac{e^{-\beta s}}{s}$, so $\mathscr{L}\{e^{\alpha t}\mathscr{U}(t-\beta)\} = \frac{e^{-\beta(s-\alpha)}}{s-\alpha}$

(b) $\mathscr{L}\{e^{\alpha t}\mathscr{U}(t-\beta)\} = e^{-\beta s}\mathscr{L}\{e^{\alpha(t+\beta)}\} = e^{-\beta s}e^{\alpha\beta}\mathscr{L}\{e^{\alpha t}\} = \frac{e^{-\beta s}e^{\alpha\beta}}{s-\alpha} = \frac{e^{-\beta(s-\alpha)}}{s-\alpha}$

Problem 6. (Challenge Problem)

(a) Since $\mathscr{U}(t-a) = 0$ for all $t < a$, we have

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = \int_0^\infty e^{-st} f(t-a)\mathscr{U}(t-a)\,dt = \int_0^a (0)\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

Let $u = t-a$ so that $du = dt$ and $t = u+a$. We have $u = 0$ when $t = a$ and $u \to \infty$ as $t \to \infty$. Moreover, $e^{-s(u+a)} = e^{-as}e^{-su}$. So

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}\int_0^\infty e^{-su} f(u)\,du = e^{-as}\mathscr{L}\{f(t)\} \quad \text{as expected.}$$

(b) The set up here is identical to that used above. The only difference is that upon substitution, $f(t) = f(u+a)$.

$$\mathscr{L}\{f(t)\mathscr{U}(t-a)\} = \int_0^\infty e^{-st} f(t)\mathscr{U}(t-a)\,dt = \int_a^\infty e^{-st} f(t)\,dt$$

Now let $u = t-a$ as before to obtain

$$\mathscr{L}\{f(t)\mathscr{U}(t-a)\} = e^{-as}\int_0^\infty e^{-su} f(u+a)\,du = e^{-as}\mathscr{L}\{f(u+a)\}.$$

Relabel the variable $u \to t$ to write it in the more familiar form $\mathscr{L}\{f(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}$.

2.16 Section 16: Laplace Transforms of Derivatives and IVPs

Go to problems 1.16

Get Acquainted Problem 16.

(a) $sY(s) - y(0) + 3Y(s) = \dfrac{1}{s^2}$

(b) $Y(s) = \dfrac{1}{s^2(s+3)} + \dfrac{1}{s+3}$

(c) $Y(s) = \dfrac{-\frac{1}{9}}{s} + \dfrac{\frac{1}{3}}{s^2} + \dfrac{\frac{10}{9}}{s+3}$ so that $y(t) = -\dfrac{1}{9} + \dfrac{1}{3}t + \dfrac{10}{9}e^{-3t}$

(d) Substitute

(e) $\mu = e^{3t}$ The equation becomes $\int \frac{d}{dt}[e^{3t}y]\,dt = \int te^{3t}\,dt = \dfrac{1}{3}te^{3t} - \dfrac{1}{9}e^{3t} + C$. Applying the initial condition given $C = \dfrac{10}{9}$. The final solution is the same as before $y(t) = -\dfrac{1}{9} + \dfrac{1}{3}t + \dfrac{10}{9}e^{-3t}$.

Problem 1.

(a) $y = \frac{1}{6}e^{t} - \frac{9}{2}e^{-t} + \frac{16}{3}e^{-2t}$

(b) $y = \frac{5}{3}\sin t - \frac{1}{3}\sin(2t)$

(c) $y = 2 - e^{-2t} + e^{t}$

(d) $y = \frac{7}{3} - \frac{7}{2}e^{-t} + \frac{1}{6}e^{3t}$

(e) $y = \frac{1}{6}t^3 + 1 - t + t^2$

Problem 2.

(a) $y = \frac{1}{2}t^2e^{3t} + e^{3t} - 3te^{3t}$

(b) $y = 2e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)$

(c) $y = \frac{1}{30}t^6e^{-4t} + e^{-4t} + 5te^{-4t}$

Problem 3.

(a) $y = (1 - \cos(t-1))\mathscr{U}(t-1)$

(b) $y = \left(e^{t-1} - t\right)\mathscr{U}(t-1) + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$

Problem 4.

(a) $y = 1 - e^{-t} - (1 - e^{-(t-3)})\mathscr{U}(t-3) = \begin{cases} 1 - e^{-t}, & 0 \le t < 3 \\ (e^3 - 1)e^{-t}, & t \ge 3 \end{cases}$

(b) $y = 3(1 - \cos t) - 3(1 - \cos(t - \pi))\mathscr{U}(t - \pi)$

Problem 5. $x(t) = 1 - \cos(4t) - [1 - \cos(4(t-2))]\,\mathscr{U}(t-2)$

Problem 6. $q(t) = \frac{1}{5}e^{-20(t-1)}\mathscr{U}(t-1)$

Problem 7. $x(t) = \cos(t), \quad y(t) = \sin(t)$

Problem 8. $x(t) = 3e^{2t} + e^{-2t}, \quad y(t) = e^{2t} - e^{-2t}$

Problem 9.

$$\begin{aligned} x(t) &= 2\cos(2t) - \frac{3}{2}\sin(2t) + 3t + 6 \\ y(t) &= \frac{3}{2}\sin(2t) - 2\cos(2t) + t + 2 \end{aligned}$$

Problem 10. $X(s) = \dfrac{F_0\omega}{(s^2+\omega^2)^2} = \dfrac{F_0}{\omega}\left(\dfrac{\omega}{s^2+\omega^2}\right)^2 = \dfrac{F_0}{\omega}\left(\mathscr{L}\left\{\sin(\omega t)\right\}\right)^2$

$$\begin{aligned} x(t) &= \frac{F_0}{\omega}\left(\sin(\omega t) * \sin(\omega t)\right)(t) \\ &= \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0 t}{2\omega}\cos(\omega t) \end{aligned} \tag{2.2}$$

2.17 Section 17: Fourier Series: Trigonometric Series

Go to problems 1.17

Get Acquainted Problem 17.

(a) $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}(\pi - x)\,dx\right) = \dfrac{\pi}{2}$

(b) $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \dfrac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}(\pi - x)\cos(nx)\,dx\right) = \dfrac{1-(-1)^n}{n^2\pi}$

(c) $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \dfrac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}(\pi - x)\sin(nx)\,dx\right) = \dfrac{1}{n}$

(d) $f(x) = \dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos(nx) + \frac{1}{n}\sin(nx)\right)$

Problem 1.

(a) The average value $f_{avg} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

(b) Going to the integral formula for a_0 and using the fact that $\pi - (-\pi) = 2\pi$ $\dfrac{a_0}{2} = \dfrac{1}{2}\left(\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx\right) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi-(-\pi)}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = f_{avg}$.

Problem 2.

(a) $f(x) = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)$

(b) $f(x) = \dfrac{\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)$

(c) $f(x) = \dfrac{2+\pi}{4} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2\pi}\cos(nx) + \frac{(1-\pi)(-1)^n - 1}{n\pi}\sin(nx)\right)$

Problem 3.

(a) $f(x) = 2 + \displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi}\sin(n\pi x)$

(b) $f(x) = \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right)$

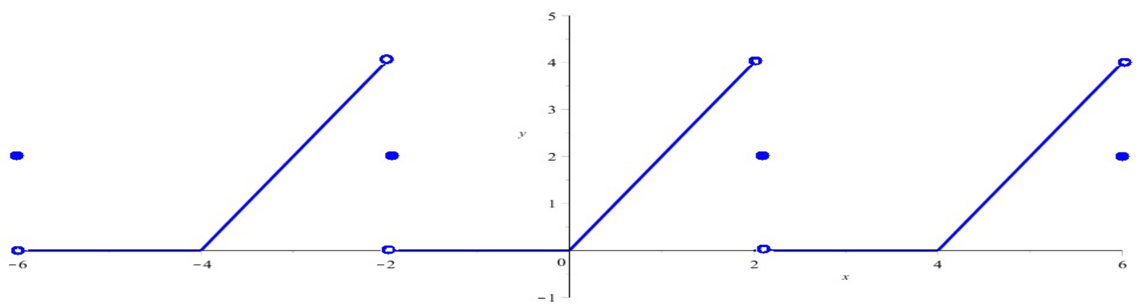
(c) $f(x) = 1 + \displaystyle\sum_{n=1}^{\infty}\left[\frac{4((-1)^n - 1)}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{4(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]$

Problem 4. This is straightforward integration. The following trigonometric identities are useful

$$\begin{aligned}
\sin(A)\cos(B) &= \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)\\
\sin(A)\sin(B) &= \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)\\
\cos(A)\cos(B) &= \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)\\
\cos^2(A) &= \frac{1}{2}(1+\cos(2A)) \quad \text{and}\\
\sin^2(A) &= \frac{1}{2}(1-\cos(2A))
\end{aligned}$$

Problem 5. Letting $S_f(x)$ denote the value to which the series converges at x, we have

$$S_f(-2) = 2, \quad S_f(-1) = 0, \quad S_f(0) = 0, \quad S_f(1) = 2, \quad S_f(2) = 2, \quad \text{and} \quad S_f(4.5) = 1$$



Fourier series of $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 2x, & 0 \le x < 2 \end{cases}$ on $[-6, 6]$.

Problem 6. **(Challenge Problem)**

(a) Setting $x = 0$ and using that $\cos(0) = 1$

$$\begin{aligned} \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(0) &= 0 \quad \Longrightarrow \\ \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} &= -\frac{\pi^2}{3} \quad \Longrightarrow \\ \sum_{n=1}^{\infty} \frac{-(-1)^n}{n^2} &= -\frac{1}{4}\left(-\frac{\pi^2}{3}\right) \quad \Longrightarrow \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} &= \frac{\pi^2}{12} \qquad \text{as expected.} \end{aligned}$$

(b) The series converges to π^2 at $x = \pi$ as this is $\frac{1}{2}(f(-\pi-) + f(\pi+))$. So taking $x = \pi$, we have

$$\begin{aligned} \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi) &= \pi^2 \quad \Longrightarrow \\ \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}(-1)^n &= \frac{2\pi^2}{3} \quad \Longrightarrow \\ \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} &= \frac{1}{4}\left(\frac{2\pi^2}{3}\right) \quad \Longrightarrow \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6} \qquad \text{as expected.} \end{aligned}$$

2.18 Section 18: Sine and Cosine Series

Go to problems 1.18

Get Acquainted Problem 18.

(a) $f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$

(b) $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{n^2\pi^2} \cos(n\pi x)$

(c) See Figure 2.1.

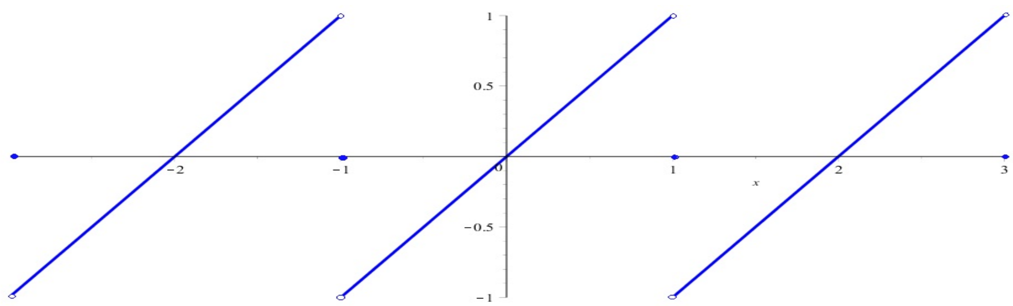


Figure 2.1: Half range sine series of $f(x) = x$ for $0 < x < 1$ showing odd extension, convergence in the mean, and 2-periodicity.

(d) See Figure 2.2.

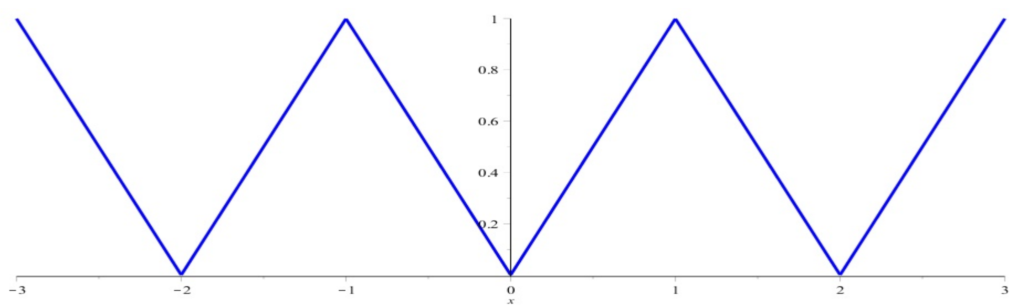


Figure 2.2: Half range cosine series of $f(x) = x$ for $0 < x < 1$ showing even extension and 2-periodicity.

Problem 1.

(a) Assume throughout that x is in $(-a,a)$. Since both are even, we have $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$ so that we conclude that the product is even.

(b) Since f is even and g is odd, we have $(fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x) = -(fg)(x)$, and we conclude that the product is odd.

(c) By the odd symmetry of f and g, $(fg)(-x) = f(-x)g(-x) = -f(x)(-g(x)) = (-1)^2 f(x)g(x) = f(x)g(x) = (fg)(x)$, and we conclude that the product is even.

Problem 2.

(a) f is even. $f(x) = \dfrac{\pi}{2} + \sum_{n=1}^{\infty} \dfrac{2((-1)^n - 1)}{n^2\pi} \cos(nx)$

(b) f is odd. $f(x) = \sum_{n=1}^{\infty} \dfrac{2}{n\pi}\left((-1)^n - \cos\left(\dfrac{n\pi}{2}\right)\right)\sin(n\pi x)$

(c) f is even. $f(x) = \dfrac{1}{4} + \sum_{n=1}^{\infty} \dfrac{4}{n^2\pi^2}\left(1 - \cos\left(\dfrac{n\pi}{2}\right)\right)\cos\left(\dfrac{n\pi x}{2}\right)$

Problem 3.

(a) $f(x) = \sum_{n=1}^{\infty}\left(\dfrac{2\pi(-1)^{n+1}}{n} + \dfrac{4}{n^3\pi}((-1)^n - 1)\right)\sin(nx)$ and

$f(x) = \dfrac{\pi^2}{3} + \sum_{n=1}^{\infty}\dfrac{4(-1)^n}{n^2}\cos(nx)$

(b) $f(x) = \sum_{n=1}^{\infty}\dfrac{8(1-(-1)^n)}{n\pi}\sin(n\pi x)$ and $f(x) = 4$

(c) $f(x) = \sum_{n=1}^{\infty}\left(\dfrac{4}{n^2\pi^2}\sin\left(\dfrac{n\pi}{2}\right) - \dfrac{2(-1)^n)}{n\pi}\right)\sin(n\pi x)$ and

$f(x) = \dfrac{3}{4} + \sum_{n=1}^{\infty}\left[\dfrac{4}{n^2\pi^2}\left(\cos\left(\dfrac{n\pi}{2}\right) - 1\right)\cos(n\pi x)\right]$

(d) $f(x) = \sum_{n=1}^{\infty}\left[\dfrac{2n\pi(e^{-2}(-1)^n - 1)}{n^2\pi^2 + 4}\sin\left(\dfrac{n\pi x}{2}\right)\right]$ and

$f(x) = \dfrac{1 - e^{-2}}{2} + \sum_{n=1}^{\infty}\left[\dfrac{4(1 - e^{-2}(-1)^n)}{n^2\pi^2 + 4}\cos\left(\dfrac{n\pi x}{2}\right)\right]$

Problem 4. (Challenge Problem)

(a) Applying the regular derivative rules for exponentials and sine/cosine, we have $\frac{\partial u_n}{\partial t} = -n^2 b_n e^{-n^2 t}\sin(nx)$ and $\frac{\partial^2 u_n}{\partial x^2} = -n^2 b_n e^{-n^2 t}\sin(nx)$. So the PDE reduces to an identity upon substitution.

(b) Recalling that $\sin(0) = 0$ and $\sin(n\pi) = 0$ for every positive integer n, we have $u_n(0,t) = b_n e^{-n^2 t}\sin(0) = 0$ and $u_n(\pi,t) = b_n e^{-n^2 t}\sin(n\pi) = 0$.

(c) Setting $t = 0$ and using the fact that $e^0 = 1$, we get

$$\sum_{n=1}^{\infty} b_n \sin(nx) = f(x) = x(\pi - x)$$

so that the b_n are given by the half range Fourier sine coefficient formula

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx = \frac{4(1-(-1)^n)}{\pi n^3}$$

Finally,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3} e^{-n^2 t}\sin(nx).$$

Chapter 3

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